The effect of excess of loss reinsurance with reinstatements on the cedent's portfolio

Jean-François Walhin (Brüssel) and *José Paris* (Louvain-la-Neuve)

1. Introduction

Let us assume a portfolio with the following characteristics:

the number of claims is a random variable N
the severity of the claims is a random variable X

Then

$$S = X_1 + \ldots + X_N$$

is the aggregate claims distribution of the portfolio for the period considered.
We will assume that the X_i's are iid and independent of N.
In order to limit the heterogeneity of his portfolio, the Insurer may ask for an excess of loss reinsurance.
In such a treaty, the Reinsurer will pay the part of every claim exceeding a deductible D up to a limit D+L. L is called the layer.
We have for each claim:

$$A_i = \begin{cases} X_i & \text{if } X_i < D \\ D & \text{if } D \le X_i \le D+L \\ X_i - L & \text{if } X_i > D+L \end{cases}$$

the retained part of a risk for the cedent. $R_i = X_i - A_i$ is the part of the risk for which the Reinsurer is liable.
The aggregate parts for both the Insurer and the Reinsurer are

$$S_A = A_1 + \ldots + A_N$$

$$S_R = R_1 + \ldots + R_N$$

Note that there may be more than one layer as it will be seen in the subsequent sections.
In this paper we will be interested in the probability of ruin in discrete time. Let U_n denotes the surplus of the insurer at time n. We have

$$U_n = u + nc - (S_1 + \ldots + S_n)$$

where u denotes the initial surplus of the insurer
c denotes the premium earned each year
S_i denotes the aggregate claims for the year i $1 \le i \le n$
We assume that the S_i are iid

It is well known that the probability of ruin is bounded (Lundberg's inequality)

$$\mathbb{P}[\text{ruin}] \le e^{-Ru} \tag{1}$$

where R, the adjustment coefficient, is given by the only strictly positive solution of

$$e^{-cr}\,\mathbb{E}[e^{rS}] = 1 \tag{2}$$

In view of (1), we are interested in getting the higher adjustment coefficient. However the Insurer is also interested in having the higher expected gain

$$\mathbb{E}G = c - \mathbb{E}S \tag{3}$$

Bowers et al. (1986) studied some reinsurance arrangements, among others the excess of loss treaty, from the point of view of the expected gain and the adjustment coefficient.

The aim of this paper is to make the same kind of study for excess of loss arrangements with reinstatements. Indeed, in practice, the Reinsurer will often limit his liability to a fixed number of times the layer.

If the layer may be consumed $k+1$ times, we say that there are k reinstatements. The latter may be free or paid. In case of paid reinstatements, the reinstatement premiums are a fraction c_i, $1 \le i \le k$ of the initial reinsurance premium p_0 in proportion of the part of the layer hit by the claims.

Sundt (1991) has given the methodology to price excess of loss treaties with reinstatements.

If there are k reinstatements in our excess of loss arrangement, the part of the cedent becomes

$$S_{Ced} = S_A + \max(0, S_R - (k+1)L) + p_0 \sum_{i=1}^{k} c_i \min\left(1, \max\left(0, \frac{S_R - (i-1)L}{L}\right)\right) \tag{4}$$

We assume implicitely that the reinstatement premiums are considered as claims because we put them in S_{Ced}.

Now, if we want to study the adjustment coefficient of the cedent's portfolio, we have to compute the distribution of S_{Ced} which is a function of S_A and S_R that are not independent. We therefore need the joint distribution of (S_A, S_R) in order to find the distribution S_{Ced}.

We have

$$(S_A, S_R) = \left(\sum_{i=1}^{N} A_i, \sum_{i=1}^{N} R_i\right)$$

By restricting the X_i's to be arithmetic and N to belong to the Panjer's class of counting distributions, we will extend Panjer's formula for this bivariate case in order to obtain the bivariate distribution of aggregate claims.

The paper is organized as follows: section 2 deals with the bivariate extension of Panjer's algorithm. Section 3 compares different reinstatements from the point of view of the cedent for given portfolios. Section 4 gives the multivariate extension of Panjer's algorithm. Section 5 completes section 3 by considering excess of loss arrangements with multiple layers. Section 6 shows how to organize the calculations. Section 7 gives the conclusion.

2. Bivariate Panjer's recursion

Bivariate extensions of Panjer's algorithm have been recently studied in Hesselager (1996) where one can find the bivariate aggregate claim distribution of (S_1, S_2) with

$$S_1 = \sum_{i=1}^{N_0+N_1} X_i$$

$$S_2 = \sum_{i=1}^{N_0+N_2} Y_i$$

where the N_i, $0 \leq i \leq 2$ belong to Panjer's class of counting distributions and the X_i are independent of the Y_i.

If one takes the particular case that N_1 and N_2 are degenerate to 0, we have almost the solution to the problem raised in the introduction behave that we have to relax the hypothesis of independence between X_i and Y_i. This is done in the following proposition.

We will adopt the following notation:

$$\sum_{x_1,\dots,x_k}^{s_1,\dots,s_k} g(x_1,\dots,x_k) = \sum_{x_1=0}^{s_1} \cdots \sum_{x_k=0}^{s_k} g(x_1,\dots,x_k) - g(0,\dots 0)$$

Let us note that the multivariate extension of Panjer's recursion has been discovered independently by Sundt (1999) and the authors of the present work. We mention the proof of the proposition because it is based on probability generating functions while Sundt's proof is based on conditional distributions.

Proposition 1

Let

$$(S_1, S_2) = \left(\sum_{i=1}^{N} X_i, \sum_{i=1}^{N} Y_i \right)$$

with *(X_i, Y_i) iid, arithmetic and independent of N*
X_i and Y_i are not necessarily independent
N belonging to Panjer's class, i.e. such that

$$\frac{p(n)}{p(n-1)} = a + \frac{b}{n}, \quad n \geq 1$$

Then, if $\Psi_N(z)$ denotes the probability generating function of N, we have

$$f_{(S_1,S_2)}(0,0) = \Psi_N\left(f_{(X_1,X_2)}(0,0)\right) \tag{5}$$

$$f_{(S_1,S_2)}(j,k) = \frac{1}{1 - a f_{(X_1,X_2)}(0,0)} \sum_{m}^{j} \sum_{n}^{k} \left[a + b\frac{m}{j} \right] \cdot f_{(S_1,S_2)}(j-m, k-n)\, f_{(X_1,X_2)}(m,n) \quad j \geq 1 \tag{6}$$

$$f_{(S_1,S_2)}(j,k) = \frac{1}{1 - a f_{(X_1,X_2)}(0,0)} \sum_{m}^{j} \sum_{n}^{k} \left[a + b\frac{n}{k} \right] \cdot f_{(S_1,S_2)}(j-m, k-n)\, f_{(X_1,X_2)}(m,n) \quad k \geq 1 \tag{7}$$

Proof: Let $\Psi_{(X_1, X_2)}(z_1, z_2)$ and $\Psi_{(S_1, S_2)}(z_1, z_2)$ be the probability generating functions of (X_1, X_2) and (S_1, S_2) respectively.

From

$$f_{(S_1,S_2)}(j,k) = \sum_{n=0}^{\infty} p(n) f^{*n}_{(X_1,X_2)}(j,k)$$

we get

$$\Psi_{(S_1,S_2)}(z_1,z_2) = \sum_{n=0}^{\infty} p(n) \Psi^{n}_{(X_1,X_2)}(z_1,z_2)$$

from which (5) follows immediately.

By hypothesis, one has

$$np(n) = a(n-1)\,p(n-1) + (a+b)\,p(n-1)$$

If $\Psi^{(z_i)}_{(X_1, X_2)}(z_1, z_2)$ is the partial derivative of $\Psi_{(X_1, X_2)}(z_1, z_2)$ with respect to z_i and if we multiply on both sides by $\Psi^{n-1}_{(X_1, X_2)}(z_1, z_2)\, z_1 \Psi^{(z_1)}_{(X_1, X_2)}(z_1, z_2)$ and sum over $n = 1 - \infty$, we get

$$\begin{aligned} z_1 \Psi^{(z_1)}_{(S_1, S_2)}(z_1, z_2) &= a \Psi_{(X_1, X_2)}(z_1, z_2)\, z_1 \Psi^{(z_1)}_{(S_1, S_2)}(z_1, z_2) \\ &\quad + (a+b)\, z_1 \Psi^{(z_1)}_{(X_1, X_2)}(z_1, z_2)\, \Psi_{(S_1, S_1)}(z_1, z_2) \end{aligned}$$

which gives, returning in the initial space

$$\begin{aligned} j f_{(S_1,S_2)}(j, k) &= a \sum_{m=0}^{j} f_{(X_1,X_2)}(m, n)\,(j-m)\, f_{(S_1,S_2)}(j-m, k-n) \\ &\quad +(a+b) \sum_{m=0}^{j} \sum_{n=0}^{k} f_{(X_1,X_2)}(m, n)\, m f_{(S_1,S_2)}(j-m, k-n) \end{aligned}$$

(6) is easily derived after an algebraic manipulation
(7) follows similarly which completes the proof. □

Of course more complicated bivariate recursions can be derived in the same way, for example with the Sundt and Jewell (1981) family of counting distributions:

$$\frac{p(n)}{p(n-1)} = a + \frac{b}{n} \quad n = w, w+1, \ldots \quad w > 1$$

Proposition 2

Let

$$(S_1, S_2) = \left(\sum_{i=1}^{N} X_i, \sum_{i=1}^{N} Y_i \right)$$

with *(X_i, Y_i) iid, arithmetic and independent of N*
X_i and Y_i are not necessarily independent
N belonging to Sundt and Jewell family of counting distributions,

Then, if $\Psi_N(z)$ denotes the probability generating function of N, we have

$$f_{(S_1,S_2)}(0,0) = \Psi_N(f_{(X_1,X_2)}(0,0))$$

$$\begin{aligned} f_{(S_1,S_2)}(j,k) &= \frac{1}{1 - a f_{(X_1,X_2)}(0,0)} \left(\sum_{m}^{j} \sum_{n}^{k} \left[a + b\frac{m}{j} \right] f_{(S_1,S_2)}(j-m, k-n) f_{(X_1,X_2)}(m,n) \right. \\ &\quad \left. + \sum_{i=1}^{w} p(i)\, f_X^{*i}(x)\, f_Y^{*i}(y) \right) \quad j \geq 1 \end{aligned}$$

$$\begin{aligned} f_{(S_1,S_2)}(j,k) &= \frac{1}{1 - a f_{(X_1,X_2)}(0,0)} \left(\sum_{m}^{j} \sum_{n}^{k} \left[a + b\frac{n}{k} \right] f_{(S_1,S_2)}(j-m, k-n) f_{(X_1,X_2)}(m,n) \right. \\ &\quad \left. + \sum_{i=1}^{w} p(i)\, f_X^{*i}(x)\, f_Y^{*i}(y) \right) \quad k \geq 1 \end{aligned}$$

3. Comparison of an excess of loss arrangement with different reinstatements

Let us take a numerical example. We assume that the claims severity is given by

Table 1. Claim severity distribution from group up

X	1	2	3	4	5	6	8	10	12	14
$f_X(x)$	0.2	0.15	0.15	0.2	0.06	0.06	0.06	0.05	0.04	0.03

The number of claims is Poisson distributed with mean $\lambda=3$.
The reinsurance layer is 4x6, i.e. the Reinsurer pays for the part of the claims exceeding 6 with a maximum payment of 4 per claim.
Let us assume that the cedent loading is 50% and the Reinsurer loading is 100%.
We calculate the premiums according to the net premium principle.
The following table give the reinsurance premium in function of the number of reinstatements (k) as well as the percentage (c_i) of the reinstatements premiums:

Table 2. Reinsurance premiums

c_i	k = 0	1	2	3
0%	1.4592	1.7550	1.7955	1.7996
50%		1.4843	1.4724	1.4697
100%		1,2859	1.2479	1.2420
150%		1.1343	1.0828	1.0754
1st: 100% 2nd: 0%			1.3155	
1st: 0% 2nd: 100%			1.6718	

These premiums are easily obtained with the methodology of Sundt (1991).
Note the interesting comportment of the reinsurance premium when the reinstatements are payable: it diminishes with the number of reinstatements for this particular example.
The following table gives the adjustment coefficient and the expected gain for the different reinsurance structures described hereabove.

Table 3. Adjustment coefficient and expected gain

c	k = 0	1	2	3
0%	0.1019	0.1142	0.1223	0.1252
50%		0.1064	0.1070	0.1065
100%		0.1008	0.0972	0.0953
150%		0.0965	0.0906	0.0880
1st: 100% 2nd: 0%			0.1064	
1st: 0% 2nd: 100%			0.1068	
$\mathbb{E}G$	4.9758	4.6799	4.6395	4.6353

The expected gain diminishes with the number of reinstatements because the loading of the Reinsurer is higher than the loading of the cedent. We can make the following comments:

1 if we have the choice only between paid reinstatements at 100%, then clearly we will choose 1 reinstatement because this situation gives the higher adjustment coefficient with the higher expected gain.
2 if we need 2 or 3 reinstatements, that are payable, we will always choose 2 for the same reason.
3 for a given number of reinstatements, that are constant, the adjustement coefficient decreases when the percentage increases.

The remarks 1 and 2 are not general and show the interest for the cedent to make the calculations in order to choose the best treaty according to the two criterions: best expected gain; best adjustment coefficient.
The remark 3 can be shown to be always true:

Proposition 3

Let

$$S_{ced,0} = S_A + \max(0, S_R - (k+1)L) + c_0\, p_0 \min\left(k, \frac{S_R}{L}\right) + p_0$$

$$S_{ced,1} = S_A + \max(0, S_R - (k+1)L) + c_1\, p_1 \min\left(k, \frac{S_R}{L}\right) + p_1$$

p_i and R_i are associated with c_i according to $S_{ced,i}$.
The p_i are calculated with the expected value principle.
If

$$c_0 < c_1$$

Then

$$R_0 > R_1$$

Proof: Let

$$g_0(S_R) | S_A = S_A + \max(0, S_R - (k+1)L) + c_0\, p_0 \min\left(k, \frac{S_R}{L}\right) + p_0$$

$$g_1(S_R) | S_A = S_A + \max(0, S_R - (k+1)L) + c_1\, p_1 \min\left(k, \frac{S_R}{L}\right) + p_1$$

$$f(S_R) = \min\left(k, \frac{S_R}{L}\right)$$

We use Ohlin's (1969) lemma with the random variable $S_R | S_A$. The hypotheses of Ohlin's lemma are verified:

$$g_0(x) | S_A \quad \text{is increasing with x}$$

$$g_1(x) | S_A \quad \text{is increasing with x}$$

$$\mathbb{E}[g_0(S_R) | S_A] = \mathbb{E}[g_1(S_R) | S_A]$$

We have

$$g_0(S_R) | S_A \leq g_1(S_R) | S_A \quad S_R \geq x_0$$

$$g_0(S_R) | S_A \geq g_1(S_R) | S_A \quad S_R \leq x_0$$

with

$$x_0 = f^{-1}\left(\frac{p_0 - p_1}{c_1 p_1 - c_0 p_0}\right)$$

Then by Ohlin's lemma we have

$$\mathbb{E}\,[e^{r g_0(S_R)} | S_A] \le \mathbb{E}\,[e^{r g_1(S_R)} | S_A]$$

Multiplying both sides by $[S_A = s_A]$ and summing on s_A gives the desired result. □

4. Multivariate extension of Panjer's recursion

When we have two layers, the aggregate claims of the cedent writes:

$$\begin{aligned}
S_{Ced} = S_A &+ \max(0, S_{R_1} - (k_1 + 1) L_1) \\
&+ \max(0, S_{R_2} - (k_2 + 1) L_2) \\
&+ p_1 \sum_{i=1}^{k_1} c_{1i} \min\left(1, \max\left(0, \frac{S_{R_1} - (i-1) L_1}{L_1}\right)\right) \\
&+ p_2 \sum_{i=1}^{k_2} c_{2i} \min\left(1, \max\left(0, \frac{S_{R_2} - (i-1) L_2}{L_2}\right)\right)
\end{aligned}$$

with

$$\begin{aligned}
R_{1i} &= \min(L_1, \max(0, X_i - D_1)) \\
R_{2i} &= \min(L_2, \max(0, X_i - (D_1 + L_1))) \\
A_i &= X_i - R_{1i} - R_{2i}
\end{aligned}$$

In practice there may be until 4 layers in an excess of loss treaty. So we need to extend the methodology to the multivariate case.
The bivariate recursion is easily extended to the following multivariate recursion using the same arguments than in section 2.

Proposition 4

Let

$$(S_1, \ldots, S_n) = \left(\sum_{i=1}^{N} X_{1i}, \ldots, \sum_{i=1}^{N} X_{ni}\right)$$

with $(X_{1i}, \ldots, X_{ni})$ *iid, arithmetic and independent of* N
The components of the vector X *are not necessarily independent*
N *belonging to Panjer's class, i.e. such that*

$$\frac{p(n)}{p(n-1)} = a + \frac{b}{n}, \quad n \ge 1$$

Then

$$f_S(0) = \Psi_N(f_X(0))$$

$$f_S(s_1,\dots,s_k) = \frac{1}{(1-a f_X(0))} \sum_{x_1,\dots,x_k}^{s_1,\dots,s_k} \left[a + b \frac{x_1}{s_1} \right] \cdot f_S(s_1 - x_1,\dots,s_k - x_k)\, f_X(x_1,\dots,x_k)\; s_1 \geq 1$$

$$\dots$$

$$f_S(s_1,\dots,s_k) = \frac{1}{(1-a f_X(0))} \sum_{x_1,\dots,x_k}^{s_1,\dots,s_k} \left[a + b \frac{x_k}{s_k} \right] \cdot f_S(s_1 - x_1,\dots,s_k - x_k)\, f_X(x_1,\dots,x_k)\; s_k \geq 1$$

5. Excess of loss arrangements with multiple layers

In this section we take the same numerical example than in section 3.
We have a second layer: 4 xs 10.
Let us assume one free reinstatement for each layer. We find:

Table 4. Reinsurance premiums, expected gain and adjustment coefficient 1

$p_1 = 3.5101$	$r = 0.1242$
$p_2 = 1.1971$	$\mathbb{E}G = 4.0813$

Now let us assume a paid reinstatement at 100% for each layer. We have three cotations. Reinsurer 1 gives the cotation by application of the net premium principle. You might observe cotation 2 and 3 on the market.

Table 5. Reinsurance premiums, expected gain and adjustment coefficient 2

	Cotation 1	Cotation 2	Cotation 3
p_1	2.5719	2.8	2.4
p_2	1.0494	0.8	1.24
p_1+p_2	3.6213	3.60	3.64
r	0.1050	0.1040	0.1057
$\mathbb{E}G$	4.0813	4.0545	4.0985

We can make the following remarks:

- the adjustment coefficient is better when the reinstatement are free; this is connected with the proposition 3.
- with the paid reinstatements, if we look only at $p_1 + p_2$ we would choose the cotation 2. However this is a nonsense because cotation 2 has the worst expected gain as well as the worst adjustment coefficient. At the opposite, the best treaty for the cedent is given by the cotation 3 even if for this cotation $p_1 + p_2$ is the highest. This shows that it is a nonsense to aggregate the reinsurance premiums of different layers when we have to compare different cotations. The structure of the agreement is such that the expected gain and the adjustment coefficient should be kept in mind.

An interesting exercise is to compare the influence of the structure of the layers. Hereunder we give the figures for the double layer structure (T1 and T2) described before as well as the figures for a simple layer structure (T: 8 xs 6). The Reinsurers applies the expected value principle. We find

Table 6. Reinsurance premiums, expected gain and adjustment coefficient 3

	2 rec at 100% on T1 1 rec at 100% on T2	2 rec free on T1 1 rec free on T2	3 rec at 100% on T1 1 rec at 100% on T2	3 rec free on T1 1 rec free on T2
p_1	2.49591	3.59103	2.48419	3.59928
p_2	1.04941	1.19715	1.04373	1.19992
p_1+p_2	3.54532	4.78818	3.52792	4.79920
r	0.10093	0.13261	0.09846	0.13568
$\mathbb{E}G$	4.04103	4.04103	4.03551	4.03550
	1 rec at 100% on T	1 rec free on T	2 rec at 100% on T	2 rec free on T
p	3.75916	4.76885	3.69682	4.79867
r	0.10783	0.13030	0.10301	0.13586
$\mathbb{E}G$	4.05149	4.05143	4.03651	4.03640

This table shows that if the reinstatements are payable at 100%, it is better for the cedent to work with one layer while with free reinstatements, a choice has to be made.

The combination of two reinstatements (paid at 100% or free) on the low layer and one on the high layer are less interesting than two reinstatements on one large layer.

Once again we remark that the reinsurance premiums are not a good criterion in order to see what is the better cover for the cedent.

6. Organization of the calculations

In this section we first study the number of multiplications required to find the multivariate distribution $(S_1, \ldots, S_n)$:

$$\sum_{x_1=0}^{s_1} \cdots \sum_{x_n=0}^{s_n} (3(\min(x_1, m_1)+1) \ldots (\min(x_n, m_n)+1)+1) \qquad (8)$$

where $(m_1, \ldots, m_n)$ are the maximal values taken by $(X_1, \ldots, X_n)$
$(s_1, \ldots, s_n)$ are the maximal values for which $(S_1, \ldots, S_n)$ are evaluated

An upper bound for (8) is

$$3(s_1+1) \ldots (s_n+1)(m_1+1) \ldots (m_n+1) \qquad (9)$$

For our trivariate example, we had

X_1 :	part of the cedent	$m_1 = 6$
X_2 :	part of the reinsurer, low layer	$m_2 = 4$
X_3 :	part of the reinsurer, high layer	$m_3 = 4$
S_A :	aggregate part for which the cedent is liable	$s_1 = 64$
S_{R1} :	aggregate part for which the first layer reinsurer is liable	$s_2 = 44$
S_{R2} :	aggregate part for which the second layer reinsurer is liable	$s_3 = 34$

The number of multiplications needed was:

Exact	46 290 821
Upper bound	53 849 250

for a precisions given by

$$\sum_{i=0}^{64}\sum_{j=0}^{44}\sum_{k=0}^{34}\mathbb{P}(S_1=i,S_{R1}=j,S_{R2}=k)=0.999999905971$$

Of course it is possible to take advantage of the particular structure of the distribution of the random vector (X_1, X_2, X_3).
For our particular case, the number of multiplications is given by

$$\sum_{x_1=1}^{s_1}[3(1+\min(x_1,m_1)+1]+\sum_{x_1=m_1}^{s_1}\sum_{x_2=0}^{s_2}[3((\min(x_1,m_1)+(1+\min(x_2,m_2)))+1]$$

$$+\sum_{x_1=m_1}^{s_1}\sum_{x_2=m_2}^{s_2}\sum_{x_3=1}^{s_3}[3((\min(x_1,m_1)+(1+\min(x_2,m_2)+(1+\min(x_3,m_3))+1]$$

An upper bound is given by

$$s_1(3m_1+4)+(s_1-m_1+1)(s_2+1)(3(m_1+m_2)+4)$$

$$+(s_1-m_1+1)(s_2-m_2+1)s_3(3(m_1+m_2+m_3)+7)$$

With this organization, the number of multiplications needed was:

Exact	4 074 885
Upper bound	4 121 732

Of course, in the particular case Poisson $(a=0)$, the recursion is simplified. The number of multiplications is given by:

$$\sum_{x_1=1}^{s_1}[3\min(x_1,m_1)+1]+\sum_{x_1=m_1}^{s_1}\sum_{x_2=1}^{s_2}[3\min(x_2,m_2)+1]$$

$$+\sum_{x_1=m_1}^{s_1}\sum_{x_2=m_2}^{s_2}\sum_{x_3=1}^{s_3}[3\min(x_3,m_3)+1]$$

An upper bound is given by

$$s_1(3m_1+1)+(s_1-m_1+1)s_2(3m_2+1)+(s_1-m_1+1)(s_2-m_2+1)s_3(3m_3+1)$$

With this organization, the number of multiplications needed was:

Exact	1 059 513
Upper bound	1 104 162

Now for our purpose that consists in evaluating the adjustment coefficient of a ceding company when it buys an excess of loss cover with reinstatements, let us remind that we only need the moment generating function of S_{Ced}.

If we restrict ourselves to a counting distribution that is Poisson distributed, we have the following property:

Let X be the distribution of the severity of claims
Let N be Poisson distributed
Let D_1 be the lower deductible we immediately have

$$S = S_1 + S_2$$
$$S = Y_1 + \ldots + Y_{N_1} + Z_1 + \ldots + Z_{N_2}$$

with $Y_1 = X \mid X \le D_1$
$N_1 = (N \mid X \le D_1)$
$Y_2 = X \mid X > D_1$
$N_2 = (N \mid X > D_1)$
$S_1 \perp S_2$

The mgf of S_{Ced} is then the product of the mgf of S_1 and of the mgf a function of $(Z_1, \ldots, Z_{N_2})$.
Then we gain time of calculation at two levels:

- The distribution of S_1 needs not to be calculated
- The multivariate distribution based on $(Z_1, \ldots, Z_{N_2})$ will be evaluated more rapidly because the mean of N_2 is lower that the one of N. Note that most of the time the difference will be very important.

If, as it was the case in our trivariate example, the high claims liable to the cedent are completely cut by the reinsurance, then the multivariate distribution based on $(Z_1, \ldots, Z_{N_2})$ is simplified because the A which is the part of the cedent is degenerated at D_1, which reduce the dimensionality of the problem.
With the following hypotheses

A : part of the cedent; degenerated at 6
R1 : part of the reinsurer, low layer; $m_2 = 4$
R2 : part of the reinsurer, high layer; $m_3 = 4$
S_A^Z : aggregate claims distribution for which the cedent is liable; $s_1 = 48 = 8 \times 6$
S_1^Z : aggregate claims distribution for which the first layer reinsurer is liable; $s_2 = 30$
S_2^Z : aggregate claims distribution for which the second layer reinsurer is liable; $s_3 = 25$

The frequency of the claims touching the reinsurance layers is $\lambda = 0.54$.
Using this technique, the number of multiplications needed was

$$\sum_{x_1=1}^{8} \sum_{x_2=0}^{s_2} \sum_{x_3=0}^{s_3} (1+\min(x_2, m_2))\,(1+\min(x_3, m_3)) + 1$$

An upper bound is given by

$$8(s_2+1)\,(s_3+1)\,((m_2+1)\,(m_3+1)+1)$$

With this organization, the number of multiplications needed was:

Exact	145 648
Upper bound	167 648

for a precision given by

$$\sum_{i=0}^{48} \sum_{j=0}^{30} \sum_{k=0}^{25} \mathbb{P}(S_1^Z = i, S_2^Z = j, S_3^Z = k) = 0.999999989229$$

7. Conclusion

In this paper we have derived a multivariate Panjer's algorithm in order to find multivariate compound distributions depending on the same counting random variable.
This algorithm has been shown to be of interest when one is interested in finding the distribution of aggregate claims for which a ceding company is liable when it buys excess of loss reinsurance with reinstatements.
The technique described authorizes the cedent to make calculations in order to compare the expected gain and the adjustment coefficient for different covers.
We remarked that

- working on the basis of the expected value principle, free reinstatements are always better than paid reinstatements
- comparing the reinsurance premiums is not a good criterion
- the technique is interesting in order to compare covers with different types of layers

Some tips have been given in order to reduce the time of calculations when the counting distribution is Poisson distributed, showing once again the great interest of the Poisson distribution for this kind of problems.

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Zusammenfassung

Die Auswirkung einer Schadenexzedenten-Rückversicherung mit Wiederauffüllung
auf das Portefeuille eines Zedenten

Der Anpassungskoeffizient für das Risiko im Eigenbehalt nach Schadenexzedenten-Rückversicherung mit Wiederauffüllung wird berechnet.
Dafür brauchen wir eine multivariable Aggregat-Schadenverteilung. Diese Verteilung wird uns durch eine multivariable Ausdehnung von Panjers Rekursion in einfacher Weise geliefert.
Numerische Beispiele zeigen den Vorteil für den Zedenten, beim Kauf einer Schadenexzedenten-Rückversicherung mit Wiederauffüllung, den Anpassungskoeffizient seines Portefeuilles zu berechnen.
Es folgt eine Diskussion über die optimale Berechnungsmethode.

Summary

The effect of excess of loss reinsurance with reinstatements on the cedent's portfolio

The adjustment coefficient for the cedent's retained risk after excess of loss reinsurance with reinstatements is calculated.
Therefore we need a multivariate aggregate claims distribution. This distribution is easily given by a multivariate extension of Panjer's recursion.
Numerical examples show the interest for the cedent to calculate the adjustment coefficient for its portfolio when buying excess of loss reinsurance with reinstatements.
An optimal organization of the calculations is discussed.