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Applied Section

# On the optimality of proportional reinsurance

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Proportional reinsurance is often thought to be a very simple method of covering the portfolio of an insurer. Theoreticians are not really interested in analysing the optimality properties of these types of reinsurance covers. In this paper, we will use a real-life insurance portfolio in order to compare four proportional structures: quota share reinsurance, variable quota share reinsurance, surplus reinsurance and surplus reinsurance with a table of lines.

*Keywords*: Proportional reinsurance; Quota share reinsurance; Variable quota share; Surplus reinsurance; Table of lines; Optimality; RORAC; de Finetti; Individual risk model

## 1. Introduction

It is well-known in literature that non-proportional reinsurance is optimal compared to proportional reinsurance. See e.g. Vermandele and Denuit [1] where it is proved that the retention of an insurer covered by an excess of loss treaty is smaller in the stop-loss order than the retention covered by any other reinsurance of the individual type (i.e. compensation on a claim by claim basis) under the hypothesis that the expected retained loss is the same in both situations as well as the loading of the reinsurer. Vermandele and Denuit [1] also show that the retention of an insurer covered by a stop-loss treaty is smaller in the stop-loss order than the retention covered by any other reinsurance treaty, under the hypothesis that the expected retained loss is the same in both situations as well as the loading of the reinsurer.

At first sight, it therefore seems that proportional reinsurance is ruled out by excess of loss and stop-loss covers, which are of the non proportional type.

In practice this is not the case for multiple reasons such as:

1. stop-loss covers are difficult to obtain due to the possible moral hazard behaviour that the ceding company may adopt after buying such a cover
2. stop-loss covers are extremely difficult to price by reinsurers

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3. the loading for a stop-loss cover will clearly differ from a proportional cover (e.g. due to the first two points)
4. excess of loss covers are sometimes difficult to price
5. the loading for an excess of loss cover will also differ from a proportional cover.

Consequently there is room for proportional covers and it is worth analysing their optimality properties.

The main objective of the present paper is to illustrate by means of a numerical example that the traditional belief that surplus treaties with a table of lines are better (more optimal) than standard surplus treaties is wrong. We will take this opportunity to compare all the proportional types of reinsurance.

The rest of the paper is organized as follows. Section 2 describes the data we will use for the numerical application. Section 3 explains how the individual risk model will be used as well as approximations of the aggregate claims distribution within the individual risk model. Section 4 describes the four types of proportional reinsurance to be compared in section 5 where we will look for optimal reinsurance structures. Section 6 concludes.

## 2. Data

For the calculations a real-life data set will be used. It is obtained from one of the leading Belgian insurance companies and contains 27 551 fire policies, covering industrial risks.

The 27 551 policies are divided into four classes ($j=1, 2, 3, 4$), depending on their claims probability ($q_{ij}$) as well as their relative claims severity ($X_{ij}$), $i=1,\dots,n_j$ where $n_j$ is the number of policies in class $j$. Knowing the sum insured $M_{ij}$, we can obtain the loss amount: $C_{ij}=M_{ij}\times X_{ij}$. We will assume the $X_{ij}$ to be identically distributed within a given risk class $(j=1,2,3,4,): X_{ij}\sim X_j,\ i=1,\dots,n_j,\ j=1,2,3,4$. We also assume that the probability of making a loss is identical within a class: $q_{ij}=q_j,\ i=1,\dots,n_j, j=1,2,3,4$.

For the density of $X_j$ we will use the MBBEFD distribution class introduced by Bernegger [2]. Using the following notations

$$b=b(c)=e^{3.1-0.15c(1+c)}$$
$$g=g(c)=e^{c(0.78+0.12c)}$$

we assume the density function of $X_j$ to be

$$f(x)=\frac{(b-1)(g-1)\ln(b)b^{1-x}}{((g-1)b^{1-x}+(1-gb))^2},\quad 0\le x<1$$
$$f(1)=\frac{1}{g}.$$

We then have a family of distributions indexed by a parameter $c$. According to Bernegger [2], $c=2, 3, 4, 5$ corresponds to the Swiss Re exposure curves 2, 3, 4 and the Lloyd's industrial exposure curve respectively. We will assume that we have the following characteristics for our portfolio:

Table 1. Claims characteristics of the portfolio.

| *Class* | *q* (%) | *c* |
|---|---|---|
| 1 | 0.75 | 2 |
| 2 | 1.00 | 3 |
| 3 | 1.25 | 4 |
| 4 | 1.50 | 5 |

Regarding the sum insured, we have the following information at disposal:

Table 2. Characteristics of the sums insured.

| *Class* (*j*) | *n* | $\mu_j(M)$ | $\sigma_j(M)$ | $\gamma_j(M)$ |
|---|---|---|---|---|
| 1 | 3933 | 13 457 022 | 10 752 926 | 8.51 |
| 2 | 17 472 | 12 034 729 | 7 960 092 | 2.23 |
| 3 | 3121 | 11 826 858 | 9 119 825 | 4.62 |
| 4 | 3025 | 10 879 648 | 7 826 747 | 11.98 |

where

$$\mu_j(M) = \frac{\sum_{i=1}^{n_j} M_{ij}}{n_j}$$

$$\sigma_j(M) = \sqrt{\frac{\sum_{i=1}^{n_j} (M_{ij} - \mu_j(M))^2}{n_j}}$$

$$\gamma_j(M) = \frac{\sum_{i=1}^{n_j} (M_{ij} - \mu_j(M))^3}{\sigma_j^3(M)}.$$

## 3. Individual risk model and approximations

Clearly our portfolio fits into the definition of the individual risk model (see e.g. Klugman, Panjer and Willmot [3]). We have $n = \sum_{j=1}^{4} n_j$ policies with a different sum insured, which are divided into four classes according to their claims behaviour (claims probability and severity).

Therefore the aggregate claims amount is given by

$$\begin{aligned} S^{ind} &= \sum_{j=1}^{4} \sum_{i=1}^{n_j} S_{ij} \\ &= \sum_{j=1}^{4} \sum_{i=1}^{n_j} D_{ij} C_{ij} \end{aligned}$$

where

1. $D_{ij}$ is the indicator function taking value 1 when there is a claim and 0 when there is no claim. We have $\mathbb{P}[D_{ij} = 1] = \mathbb{P}[D_j = 1] = q_j$.
2. $C_{ij} = M_{ij} X_{ij}$ is the conditional loss value.
3. $S_{ij} = D_{ij} C_{ij}$ is the loss associated to policy *ij*.

We will assume the mutual independence between the random variables $D_{ij}$ and $C_{ij}$.

Obtaining the exact distribution of $S^{ind}$ is possible by using recursive formulae (see e.g. Dhaene and Vandebroek [4]) but the computing time will be very long due to the size of the portfolio. Moreover a discretization of distribution of $C_{ij}$ is required.

An approximation of the individual risk model is provided by the collective risk model (see e.g. Klugman, Panjer and Willmot [3]) leading to the use of the Panjer recursive formula (see Panjer [5]). Once again the computing time will be long and discretization will be required.

In this paper, as the size of the porfolio is high, and its skewness less than 2 (see further for the calculations) we will concentrate on a parametric approximation, namely the shifted gamma distribution, that will reproduce the first three moments of the original distribution. We therefore need to obtain the first three moments of $S^{ind}$.

The shifted gamma distribution ($S$) (see e.g. Dufresne and Niederhauser [6]) has the form

$$S = Z + x_0$$

where $Z \sim Gam(\alpha, \beta)$, i.e. $Z$ has density function $f_Z(x)$ and distribution function $F_Z(x)$:

$$f_Z(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \quad x > 0$$

$$F_Z(x) = \int_0^x f_z(s)ds$$

where $\Gamma(x)$ is the gamma function. By abuse of notation, we will also write $F(\alpha, \beta, x)$ the distribution function of $Z$.

Centered moments are given by

$$\mu = \sum_{j=1}^{4} [q_j \mathbb{E}X_j] \sum_{i=1}^{n_j} M_{ij}$$

$$\mu_2 = \sum_{j=1}^{4} \left[q_j \mathbb{V}arX_j + q_j(1-q_j)(\mathbb{E}X_j)^2\right] \sum_{i=1}^{n_j} M_{ij}^2$$

$$\mu_3 = \sum_{j=1}^{4} \left[q_j \mathbb{E}X_j^3 - 3q_j^2 \mathbb{E}X_j \mathbb{E}X_j^2 + 2q_j^3(\mathbb{E}X_j)^3\right] \sum_{i=1}^{n_j} M_{ij}^3.$$

Using numerical integration, it is possible to obtain the first three moments of $X_j$, in function of the parameter $c$:

Table 3. Moments of $X_j$.

| *Class* | $\mathbb{E}X_j$ | $\mathbb{E}X_j^2$ | $\mathbb{E}X_j^3$ |
|---|---|---|---|
| 1 | 0.2260909 | 0.1623865 | 0.1474579 |
| 2 | 0.0871796 | 0.0479373 | 0.0407141 |
| 3 | 0.031852 | 0.0123161 | 0.0094975 |
| 4 | 0.0121457 | 0.0030479 | 0.0020178 |

An analytical formula exists for $\mathbb{E}X : \mathbb{E}X = \frac{\ln(gb)(1-b)}{\ln(b)(1-gb)}$ but not for higher moments.

The $\sum_{i=1}^{n_j} M_{ij}^y, y = 1, 2, 3$ terms are easily obtained from table 2.

From this we can obtain the mean $(\mu)$, the standard deviation $(\sigma)$, the coefficient of variation $\left(CV = \frac{\sigma}{\mu}\right)$ and the skewness $\left(\gamma = \frac{\mathbb{E}(S^{ind} - \mu)^3}{\sigma^3}\right)$ of $S^{ind}$:

$$\begin{aligned} \mu &= 293\ 751\ 934 \\ \sigma &= 57\ 364\ 022 \\ CV &= 0.20 \\ \gamma &= 0.6. \end{aligned}$$

The corresponding shifted gamma approximation has the following parameters:

$$\begin{aligned} \alpha &= \frac{4}{\gamma^2} = 10.44 \\ \beta &= \frac{2}{\gamma\sigma} = 5.63\ 10^{-8} \\ x_0 &= \mu - \frac{2\sigma}{\gamma} = 108\ 404\ 392. \end{aligned}$$

## 4. Proportional reinsurance

Proportional reinsurance is the easiest way of covering an insurance portfolio. In proportional reinsurance, the ceding company and the reinsurer agree on a cession percentage, say $\tau_{ij}$, for each policy in portfolio. The premium corresponding to the policy *ij,* say $P_{ij}$, is then shared proportionally between the insurer and the reinsurer. The reinsurer receives $\tau_{ij}P_{ij}$ whereas the insurer keeps the premium $(1 - \tau_{ij})P_{ij}$. If $S_{ij}$ is a claim hitting policy *ij,* the reinsurer is liable for $\tau_{ij}S_{ij}$ whereas the insurer retains $(1 - \tau_{ij})S_{ij}$.

Clearly the way a proportional reinsurance works is extremely simple. Moving to the way of fixing the cession percentage $\tau_{ij}$, we can distinguish between four subtypes of proportional reinsurance: quota share reinsurance, variable quota share reinsurance, surplus reinsurance and surplus reinsurance with a table of lines.

Note that proportional reinsurance is sometimes called pro-rata reinsurance.

We will use the following notations:

- $S_{ij}$ is the loss associated with policy *ij*.
- $S = \sum_{j=1}^{4} \sum_{i=1}^{n_j} S_{ij}$ is the aggregate loss of the insurer.
- $S^{Re} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} \tau_{ij} S_{ij}$ is the aggregate liability of the reinsurer.
- $S^R = \sum_{j=1}^{4} \sum_{i=1}^{n_j} (1 - \tau_{ij}) S_{ij}$ is the aggregate loss in retention when a reinsurance cover is bought.
- $P_{ij}$ is the premium associated with policy *ij.*
- $P = \sum_{j=1}^{4} \sum_{i=1}^{n_j} P_{ij}$ is the total premium of the insurer.
- $P^{Re} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} \tau_{ij} P_{ij}$ is the total ceded premium.
- $P^R = \sum_{j=1}^{4} \sum_{i=1}^{n_j} (1 - \tau_{ij}) P_{ij}$ the total retained premium.

It is clear that only the risk premium has to be ceded. In practice the insurer cedes on the basis of the commercial premium and the reinsurer pays a reinsurance commission representing the management expenses and acquisition costs of the ceding company. To keep things simple, we will always refer to the risk premium in the following and not to the reinsurance commission.

### *4.1. Quota share reinsurance*

In quota share reinsurance $\tau_i$ is the same for the whole insurance portfolio. Quota share reinsurance is therefore extremely simple as the cession percentage does not vary among policies: we note it as $\tau$. As a consequence the administration of a quota share treaty is straightforward: it suffices to obtain the total premium and the total claims in order to share the premium and the claims with the reinsurer. In particular administrative costs for handling quota share treaties are low. Quota share reinsurance is of the individual type and of the global type at the same type:

$$S^{Re} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} \tau_{ij} S_{ij} = \tau \sum_{j=1}^{4} \sum_{i=1}^{n_j} S_{ij} = \tau S.$$

Quota share reinsurance has a nice property if we compare its use to the use of the allocated capital ($u$).

Let $\epsilon$ be the ruin probability without quota share reinsurance:

$$\epsilon = \mathbb{P}[S > u + P].$$

Let $\epsilon_R$ be the ruin probability after quota share reinsurance:

$$\begin{aligned}\epsilon_R &= \mathbb{P}[(1-\tau)S > u + (1-\tau)P] \\ &= \mathbb{P}\left[S > \frac{u}{1-\tau} + P\right] \\ &< \epsilon.\end{aligned}$$

We observe that buying a quota share treaty has the same effect as increasing the economic capital in the same proportion as the cession percentage.

Now let us analyse the retained risk of a portfolio covered by a quota share treaty:

$$\begin{aligned}\mathbb{E}S^R &= (1-\tau)\mathbb{E}S \\ \mathbb{V}arS^R &= (1-\tau)^2\mathbb{V}arS \\ \sigma(S^R) &= (1-\tau)\sigma(S) \\ CV(S^R) &= CV(S) \\ \mathbb{E}(S^R - \mathbb{E}S^R)^3 &= (1-\tau)^3\mathbb{E}(S - \mathbb{E}S)^3 \\ \gamma(S^R) &= \gamma(S).\end{aligned}$$

Here we can observe that the variability and the skewness of the retained risk is the same as if there was no quota share reinsurance. Obviously quota share reinsurance does not provide a reduction in the relative homogeneity of the portfolio.

It is nevertheless very much used for multiple reasons such as

- Financing management and acquisition costs by means of the reinsurance commission (in case of a new product or a start-up insurance company).
- Reinsurance against underpricing (new classes of business). It limits the danger of new(unknown) risks.
- Reduction of the required solvency margin.
- Compensation for less balanced treaties of the cedant.

Note that quota share reinsurance is sometimes referred to as participating reinsurance.

### *4.2. Variable quota-share reinsurance*

Sometimes, the cession percentage may vary within the portfolio. This is called variable quota share reinsurance. In our example, we will assume that the percentage may vary in function of the class of risk. This is equivalent to analysing four different quota share treaties.

Defining $S_j = \sum_{i=1}^{n_j} S_{ij}$, we then have the following relations:

$$\mathbb{E}S^R = \sum_{j=1}^{4}(1-\tau_j)\mathbb{E}S_j$$

$$\mathbb{V}arS^R = \sum_{j=1}^{4}(1-\tau_j)^2\mathbb{V}arS_j$$

$$\sigma(S^R) = \sqrt{\sum_{j=1}^{4}(1-\tau_j)^2\mathbb{V}arS_j}$$

$$CV(S^R) \neq CV(S)$$

$$\mathbb{E}(S^R - \mathbb{E}S^R)^3 = \sum_{j=1}^{4}(1-\tau_j)^3\mathbb{E}(S_j - \mathbb{E}S_j)^3$$

$$\gamma(S^R) \neq \gamma(S)$$

It becomes impossible to compare the coefficient of variation and the skewness analytically. This will be done numerically.

### *4.3. Surplus reinsurance*

In surplus reinsurance the cession percentage is a function of both the sum insured and the line, or retention, chosen by the ceding company.

The line ($R$) is the maximal amount that the insurer wants to pay in case of a loss. If one wants to make use of proportional reinsurance and of the property that the maximal loss will never be larger than the line, the cession percentage must be defined as

$$\tau_{ij} = \max\left(0, 1 - \frac{R}{M_{ij}}\right).$$

The retention percentage is

$$1 - \tau_{ij} = \min\left(1, \frac{R}{M_{ij}}\right).$$

In case of a total loss, the retained loss is

$$\min\left(1, \frac{R}{M_{ij}}\right) \times M_{ij} = \begin{cases} M_{ij} \text{ if } M_{ij} < R \\ \frac{R}{M_{ij}} M_{ij} = R \text{ if } M_{ij} > R. \end{cases}$$

It is clear that surplus reinsurance is appealing from an optimality point of view. In surplus reinsurance, the loss amount may not exceed the line. Furthermore, the smallest risks are not reinsured. Therefore, one feels that the retained risk will be more homogeneous than it is in case of a quota share reinsurance. The retained risk has the following centered moments:

$$S^R = \sum_{j=1}^{4} \sum_{i=1}^{n_j} (1 - \tau_{ij}) D_{ij} C_{ij}$$

$$\mu = \sum_{j=1}^{4} \left[q_j \mathbb{E}X_j\right] \sum_{i=1}^{n_j} (1 - \tau_{ij}) M_{ij}$$

$$\mu_2 = \sum_{j=1}^{4} \left[q_j \mathbb{V}ar X_j + q_j(1 - q_j)(\mathbb{E}X_j)^2\right] \sum_{i=1}^{n_j} (1 - \tau_{ij})^2 M_{ij}^2$$

$$\mu_3 = \sum_{i=1}^{4} \left[q_j \mathbb{E}X_j^3 - 3q_j^2 \mathbb{E}X_j \mathbb{E}X_j^2 + 2q_j^3 (\mathbb{E}X_j)^3\right] \sum_{i=1}^{n_j} (1 - \tau_{ij})^3 M_{ij}^3.$$

It is not possible to make analytical comparisons with these formulae. We will therefore concentrate on the numerical application in order to make further comments.

One should note that surplus reinsurance is far more expensive from an administrative point of view since each policy must be scanned in order to compute the ceded premium and the possible recovery from the reinsurer, based on its own cession percentage, which is a function of the insured sum.

Note that surplus reinsurance is sometimes referred to as surplus share reinsurance.

### *4.4. Surplus reinsurance with a table of lines*

We now move to surplus reinsurance with a table of lines. In the above definition of surplus reinsurance, the same retention $R$ is used for the whole portfolio. In practice however, it may happen that a surplus programme is presented with a table of lines. This means that a retention is fixed per group of similar risks. In this way the portfolio that the ceding company keeps in retention is qualitatively more homogeneous. It is especially the fire risks in an insurer's portfolio that may differ in quality. Determining factors are the location of the risk, the building's construction, its use, the loss prevention and protection measures that are taken, ... The quality of the risk is translated into a claims and severity probability: the better the risk, the smaller the claims probability and the less dangerous the claims severity. So we have four classes of risks with different characteristics. If we choose the same retention for the entire portfolio as we described above, the average loss per risk would not be homogeneous. With the same retention the yearly average loss of the ceding company would depend upon the kind of risk that has been affected. We therefore choose a different retention per class, in order to make the average loss per risk independent of the kind of risk. As a consequence the insurer is able to retain more of the good risks and less of the bad risks. For the reinsurer however there is always the risk that only the dangerous risks are transferred. When the cedant's tariff is wrong, this implies a danger to the reinsurer. This phenomenon is called antiselection.

For our numerical application, we will have

$$\tau_{ij} = \max\left(0, 1 - \frac{R_j}{M_{ij}}\right)$$

i.e. there will be four possible lines, in function of the class of risk. This is equivalent to working with four different surplus treaties.

In order to fix the lines, practitioners use one of the following methods without any theoretical justification.

A first method to construct a table of lines is to determine a retention for each class of business by aiming at an equal maximum loss throughout the entire portfolio. This means that the lines will be such that

$$R_1 \times q_1 = R_2 \times q_2 = R_3 \times q_3 = R_4 \times q_4.$$

This is the method of the inverse claim probability.

A second method takes into account not only the claims probability but also the claims severity. This table of lines is constructed in order to reach the same average loss for all policies, contrary to the same maximum loss of the first method. This means that the lines will be such that

$$R_1 \times rate_1 = R_2 \times rate_2 = R_3 \times rate_3 = R_4 \times rate_4.$$

where $rate_j = q_j\, \mathbb{E}X_j$, $j = 1, \dots, 4$. This is the method of the inverse rate.

## 5. Optimal reinsurance

In this section we will compare the original portfolio with the retained portfolio after a proportional cession of the four types described in the previous sections. We will use the RORAC criterion, i.e. we will maximize the return on risk adjusted capital of the retained risk.

We will assume that the insurer is using a loading $\xi$. That loading only contains the capital charge. All administrative expenses must be charged on top of that loading. We will also assume that the reinsurer is using a loading $\xi^{Re}$. That loading includes the capital charge of the reinsurer as well as the administrative expenses. It is clear that the insurer pays for the administrative expenses of the reinsurer in the reinsurance premium. For the numerical application, we will use $\xi = 5\%$ and $\xi^{Re} = 7\%$.

Now let us compute the RORAC (Return on Risk Adjusted Capital) for different reinsurance structures.

Let us define the retained risk of the cedant: $S^R = S - S^{Re}$.

Let us assume that the required solvency level, $RSL$, is given by the Tail Value at Risk at the level $\epsilon = 99\%$.

Using our shifted gamma approximation, we have

$$\begin{aligned} RSL &= \mathbb{E}\left[S^R | S^R > VaR_{S^R}(\epsilon)\right] \\ &= \mathbb{E}[Z|Z > VaR_Z(\epsilon)] + x_0 \\ &= \frac{\alpha}{\beta}\frac{1}{1-\epsilon}(1 - F(\alpha+1, \beta, VaR_Z(\epsilon))) + x_0 \end{aligned}$$

where $VaR_Z(\epsilon) = F^{-1}(\alpha, \beta, \epsilon)$.

The retained premium is equal to

$$P^R = (1+\xi)\mathbb{E}S - (1+\xi^{Re})\mathbb{E}S^{Re}.$$

The risk adjusted capital is obtained by deducting the retained premium from $RSL$. In other words, the risk adjusted capital is the required solvency level minus the premium that is borrowed from the policyholders plus the premium that is charged by the reinsurers:

$$RAC = RSL - P^R$$

and RORAC is defined as

$$RORAC = \frac{P^R - \mathbb{E}S^R}{RAC}.$$

For the original (i.e. before any reinsurance) portfolio, we obtain the following:

$$\begin{aligned} \mathbb{E}S = \mathbb{E}S^R &= 293\ 751\ 934 \\ CV &= 0.20 \\ \gamma &= 0.62 \\ VaR &= 452\ 547\ 891 \\ RSL = TVaR &= 483\ 141\ 978 \\ P = P^R &= 308\ 439\ 531 \\ RAC &= 174\ 702\ 447 \\ RORAC &= 8.41\%. \end{aligned}$$

Now let us analyse surplus reinsurance for different lines (table 4).

We observe that the optimal line is about 20 000 000 providing a $RORAC = 10.06\%$, instead of 8.41% without reinsurance. We also observe that when the line is small, the RORAC is smaller than in the situation without reinsurance. This is due to the fact that the loading of the reinsurer is higher than the loading of the insurer. Therefore a large cession (in other words a small line) is penalized.

Table 4. RORAC in function of the line of a surplus treaty.

| *Case* | *Line* | *CV* | $\gamma$ | *RORAC* | Expected gain | $\frac{\mathbb{E}S^{Re}}{\mathbb{E}S}$ |
|---|---|---|---|---|---|---|
| 1 | 5 000 000 | 0.16 | 0.24 | 4.16% | 2 080 641 | 61.31% |
| 2 | 7 500 000 | 0.16 | 0.24 | 7.58% | 5 222 627 | 46.03% |
| 3 | 10 000 000 | 0.16 | 0.25 | 9.05% | 7 714 795 | 33.91% |
| 4 | 12 500 000 | 0.17 | 0.26 | 9.71% | 9 583 949 | 24.82% |
| 5 | 15 000 000 | 0.17 | 0.28 | 9.98% | 10 961 666 | 18.12% |
| 6 | 17 500 000 | 0.17 | 0.29 | 10.06% | 11 973 326 | 13.20% |
| 7 | 20 000 000 | 0.18 | 0.30 | 10.06% | 12 715 622 | 9.59% |
| 8 | 22 500 000 | 0.18 | 0.31 | 10.00% | 13 266 739 | 6.91% |

For each reinsurance cover, we also give the expected gain of the cedant as well as the cession percentage $\left(\frac{\mathbb{E}S^{Re}}{\mathbb{E}S}\right)$. Because the loadings of the reinsurer do not vary across the classes of business, there is a one-to-one correspondence between these figures.

Now let us analyse the RORAC for quota share treaties. We choose the cession rate so as to get the same global cession as in table 4.

Table 5. RORAC in function of the cession of a quota share treaty.

| *Case* | $\tau$ | *CV* | $\gamma$ | *RORAC* | $\frac{\mathbb{E}S^{Re}}{\mathbb{E}S}$ |
|---|---|---|---|---|---|
| 1 | 61.31% | 0.20 | 0.62 | 2.92% | 61.31% |
| 2 | 46.03% | 0.20 | 0.62 | 5.38% | 46.03% |
| 3 | 33.91% | 0.20 | 0.62 | 6.57% | 33.91% |
| 4 | 24.82% | 0.20 | 0.62 | 7.22% | 24.82% |
| 5 | 18.12% | 0.20 | 0.62 | 7.61% | 18.12% |
| 6 | 13.20% | 0.20 | 0.62 | 7.86% | 13.20% |
| 7 | 9.59% | 0.20 | 0.62 | 8.02% | 9.59% |
| 8 | 6.91% | 0.20 | 0.62 | 8.14% | 6.91% |

We confirm that the coefficient of variation and the skewness are not affected by quota share reinsurance. We observe that RORAC increases when the cession decreases and never reaches the RORAC without reinsurance. Once again this is due to the higher reinsurance loading. In such a case, quota share reinsurance cannot be optimal.

Now let us analyse the RORAC for surplus treaties with a table of lines. We choose the method of the inverse rate and we choose the lines so as to get the same global cession as in table 5.

Table 6. RORAC in function of the table of lines (inverse rate method).

| *Case* | $R_1$ | $R_2$ | $R_3$ | $R_4$ | *CV* | $\gamma$ | *RORAC* | $\frac{\mathbb{E}S^{Re}}{\mathbb{E}S}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 792 144 | 5 430 844 | 11 891 468 | 25 987 731 | 0.16 | 0.29 | 4.06% | 61.31% |
| 2 | 4 373 473 | 8 506 598 | 18 626 192 | 40 705 865 | 0.16 | 0.28 | 7.47% | 46.03% |
| 3 | 6 066 679 | 11 799 959 | 25 837 392 | 56 465 292 | 0.16 | 0.28 | 8.93% | 33.91% |
| 4 | 7 857 669 | 15 283 513 | 33 465 040 | 73 134 831 | 0.17 | 0.29 | 9.58% | 24.82% |
| 5 | 9 739 358 | 18 943 481 | 41 478 968 | 90 648 548 | 0.17 | 0.30 | 9.86% | 18.12% |
| 6 | 11 697 749 | 22 752 639 | 49 819 564 | 108 876 170 | 0.17 | 0.31 | 9.96% | 13.20% |
| 7 | 13 736 088 | 26 717 298 | 58 500 649 | 127 847 900 | 0.18 | 0.32 | 9.96% | 9.59% |
| 8 | 15 858 279 | 30 845 054 | 67 538 854 | 147 600 082 | 0.18 | 0.33 | 9.92% | 6.91% |

We observe that the RORAC is smaller in case of a table of lines than it is in case of a classical surplus with one fixed line. This is in contradiction with the practitioner's belief that a table of lines based on the inverse rate method is optimal.

Note that the table of lines based on the inverse claim probability gives worst results. This is not surprising because that method does not take into account the claim severity at all.

Obviously we could resort to numerical optimization in order to obtain the best table of lines. We prefer deriving a formula which is based on de Finetti's [7] result.

de Finetti [7] minimized the variance of the gain of the retained portfolio subject to a given expected gain. The gain of the retained portfolio is

$$Z(\tau)=\sum_{j=1}^{4}\sum_{i=1}^{n_j}\left((1+\xi_{ij})\mathbb{E}S_{ij}-\left(1+\xi_{ij}^{Re}\right)\tau_{ij}\mathbb{E}S_{ij}-(1-\tau_{ij})S_{ij}\right)$$

where $\tau$ is the vector of cession percentages $\{\tau_{11},\dots,\tau_{n_1 1,}\tau_{12},\dots\tau_{n_2 2}\tau_{13},\dots\tau_{n_3 3},$ $\tau_{14},\dots\tau_{n_4 4}\}$. The de Finetti problem is the following:

$$\min_{\tau}\mathbb{V}arZ(\tau)$$

under the constraint that

$$\mathbb{E}Z(\tau)=k.$$

de Finetti [7] showed that the solution is given by

$$\tau_{ij}=\max\left(0,1-\frac{b\xi_{ij}^{Re}\mathbb{E}S_{ij}}{\mathbb{V}arS_{ij}}\right),\quad j=1,\dots,4,\quad i=1,\dots,n_j,$$

where $b$ is a constant given by the condition $\mathbb{E}Z(\tau)=k$.

For a surplus treaty with table of lines, Glineur and Walhin [8] have used convex optimization to prove that the optimal lines are

$$R_j=\frac{\lambda\sum_{i=1}^{n_j}\xi_{ij}^{Re}\mathbb{E}[D_{ij}C_{ij}]S_{ij}}{2\sum_{i=1}^{n_j}\mathbb{V}ar[D_{ij}C_{ij}]},\quad j=1,2,3,4$$

where $\lambda$ is a constant given by $\mathbb{E}Z(\tau)=k$.

The associated cession rates are

$$\tau_{ij}=\min\left(1,\max\left(0,1-\frac{R_j}{M_{ij}}\right)\right).$$

On the reasonable assumption that the $X_{ij}$ and $D_{ij}$ are identically distributed within the class $j$ and that the reinsurance loading is the same for each risk within the class $j$, the formula is reduced to

$$R_j = \frac{\lambda \xi_j^{Re} \mathbb{E}[D_j X_j]}{2\mathbb{V}ar[D_j X_j]}, \quad j = 1, 2, 3, 4$$

where $\lambda$ is a constant given by $\mathbb{E}Z(\tau) = k$.

Using this methodology we obtain

Table 7. RORAC in function of the table of lines (de Finetti's optimal table).

| *Case* | $R_1$ | $R_2$ | $R_3$ | $R_4$ | *CV* | $\gamma$ | *RORAC* | $\frac{\mathbb{E}S^{Re}}{\mathbb{E}S}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 3 949 974 | 5 155 430 | 7 327 325 | 11 286 824 | 0.15 | 0.24 | 4.22% | 61.31% |
| 2 | 6 007 752 | 7 841 203 | 11 144 567 | 17 166 807 | 0.16 | 0.25 | 7.68% | 46.03% |
| 3 | 8 113 889 | 10 590 093 | 15 051 518 | 23 184 974 | 0.16 | 0.26 | 9.15% | 33.91% |
| 4 | 10 247 187 | 13 374 433 | 19 008 852 | 29 280 752 | 0.17 | 0.27 | 9.79% | 24.82% |
| 5 | 12 397 936 | 16 181 549 | 22 998 558 | 35 426 392 | 0.17 | 0.28 | 10.05% | 18.12% |
| 6 | 14 573 268 | 19 020 751 | 27 033 868 | 41 642 281 | 0.17 | 0.29 | 10.13% | 13.20% |
| 7 | 16 743 363 | 21 853 117 | 31 059 460 | 47 843 201 | 0.18 | 0.30 | 10.11% | 9.59% |
| 8 | 18 964 227 | 24 751 746 | 35 179 233 | 54 189 193 | 0.18 | 0.31 | 10.05% | 6.91% |

This method of building up a table of lines is more optimal than the two methods of practitioners. Note that in our numerical example, it becomes more optimal than the surplus with a single line.

For a variable quota share, Glineur and Walhin [8] have obtained the following formula

$$\tau_{ij} = \tau_j = \min\left(1, \max\left(0, 1 - \frac{\lambda \sum_{i=1}^{n_j} \xi_{ij} \mathbb{E}S_{ij}}{2\sum_{i=1}^{n_j} \mathbb{V}arS_{ij}}\right)\right)$$

where $\lambda$ is a constant given by $\mathbb{E}Z(\tau) = k$.

With this methodology, we obtain

Table 8. RORAC in function of optimal variable quota share cession rates (de Finetti method).

| *Case* | $\tau_1$ | $\tau_2$ | $\tau_3$ | $\tau_4$ | *CV* | $\gamma$ | *RORAC* | $\frac{\mathbb{E}S^{Re}}{\mathbb{E}S}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 74.71% | 57.94% | 45.15% | 3.49% | 0.19 | 0.54 | 3.13% | 61.31% |
| 2 | 64.24% | 40.51% | 22.44% | 0.00% | 0.19 | 0.51 | 5.82% | 46.03% |
| 3 | 55.89% | 26.63% | 4.33% | 0.00% | 0.19 | 0.50 | 7.11% | 33.91% |
| 4 | 49.29% | 15.64% | 0.00% | 0.00% | 0.19 | 0.49 | 7.81% | 24.82% |
| 5 | 44.30% | 7.35% | 0.00% | 0.00% | 0.19 | 0.49 | 8.23% | 18.12% |
| 6 | 40.64% | 1.26% | 0.00% | 0.00% | 0.19 | 0.49 | 8.49% | 13.20% |
| 7 | 31.39% | 0.00% | 0.00% | 0.00% | 0.19 | 0.51 | 8.59% | 9.59% |
| 8 | 22.62% | 0.00% | 0.00% | 0.00% | 0.19 | 0.53 | 8.59% | 6.91% |

Results are obviously better than for a quota share treaty. However they do not compare favourably with surplus treaties.

## 6. Conclusion

We have analysed the optimality properties of an insurance portfolio covered by a proportional reinsurance. The numerical application has confirmed that quota share reinsurance is suboptimal when compared to all other types of proportional reinsurance. In fact, quota share reinsurance will only be of interest to the ceding company when the loading of the reinsurer is smaller than the loading of the insurer. This is possible if one refers to the diversification possibilities that are offered to the reinsurer. So one may argue that less capital needs to be remunerated with the reinsurer's position. On the other hand, one may argue that the reinsurer's shareholders may require a higher cost of capital due to the agency costs that apply when underwriting a business that is less known than in the situation of the insurer. This means that ceding companies should provide as much information as possible to reinsurers in order to reduce these agency costs.

We have also observed that surplus reinsurance with a table of lines based on the inverse probability method, or inverse rate method, is not, in our numerical example, optimal when compared to surplus reinsurance with one single line. This goes against the traditional belief of practitioners. Obviously we have not proved that it is always valid but we have shown however that a table of lines is not always optimal.

On the other hand we have used the optimal table of lines using the de Finetti's criterion. This table of lines is more optimal, in our numerical example, than the other proportional reinsurance programmes.

Eventually, one should note that the reinsurer's loading would most probably not remain constant in case of surplus treaties with increasing retentions. Indeed, when increasing the retentions, the reinsured business becomes less and less balanced, implying a larger volatility for the reinsurer. Clearly the reinsurer will apply higher capital charges in these cases. Moreover the fixed management expenses of the reinsurer will be more important in those treaties where the cession is small. One therefore has to be cautious with the previous conclusions and always ask quotes from the reinsurer when analysing the optimality of a reinsurance programme.

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## References

[1] Denuit, M. and Vermandele, C., 1998, Optimal reinsurance and stop-loss order. *Insurance: Mathematics and Economics*, **22**, 229–233.
[2] Bernegger, S., 1997, The Swiss Re Exposure curves and the MBBEFD distributions class. *Astin Bulletin*, **27**, 99–111.
[3] Klugman, S.A., Panjer, H.H. and Willmot, G.E., 1998, *Loss Models* (Wiley).
[4] Dhaene, J. and Vandebroek, M., 1995, Recursions for the individual model. *Insurance: Mathematics and Economics*, **16**, 31–38.
[5] Panjer, H.H., 1981, Recursive evaluation of a family of compound distributions. *Astin Bulletin*, **12**, 22–26.

[6] Dufresne, F. and Niederhauser, E., 1997, Some analytical approximations of stop-loss premiums. *Bulletin of the Swiss Association of Actuaries*, 25–47.

[7] de Finetti, B., 1940, Il problema dei pieni. *Giorn Inst Ital Attuari*, **11**, 1–88.

[8] Glineur, F. and Walhin, J.F., 2004, de Finetti's retention problem for proportional reinsurance revisited, Unpublished manuscript.