# AN ACTUARIAL ANALYSIS OF THE FRENCH BONUS-MALUS SYSTEM

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## Abstract

The bonus-malus system in force in France differs from most of those used in industrialized countries around the world. Policyholders do not move inside a scale but their premium is obtained with the help of multiplicative CRM coefficients (CRM stands for the acronym of the French *coefficient de réduction-majoration*). The French bonus-malus system has been the topic of very few scientific investigations in the actuarial literature. This paper purposes to analyze this bonus-malus system in details. Despite its apparent simplicity, it will be seen that it leads to nontrivial mathematical problems. The financial equilibrium of the bonus-malus system is also investigated thanks to the multivariate De Pril's algorithm for the convolution of independent and identically distributed random vectors.

*Key words and phrases:* Bonus-Malus system, Markov chains, Financial stability, Convolution of independent and identically distributed random vectors

# 1 Introduction and motivation

Many important factors cannot be taken into account *a priori* when pricing motor third party liability insurance products. For instance, swiftness of reflexes, aggressiveness behind the wheel or knowledge of the highway code are difficult to integrate into risk classification. Consequently, tariff cells are still quite heterogeneous despite the use of many classification variables. This residual heterogeneity typically causes overdispersion: data involving claim counts exhibit variability exceeding that explained by Poisson models. This phenomenon can be modelled by a random effect in a statistical model.

It is reasonable to believe that the hidden characteristics are partly revealed by the number of claims reported by the policyholders. Hence the adjustment of the premium on the basis of the individual claims experience in order to restore fairness among policyholders. In that respect, the allowance of past claims in a rating model derives from an exogeneous explanation of serial correlation for longitudinal data. In this case, correlation is only apparent and results from the revelation of hidden features in the risk characteristics.

Credibility theory can be seen as the art to combine different collections of data to obtain an accurate overall estimate. In motor insurance, credibility techniques can be used to reevaluate the annual expected claim frequency given the past claim record. Bayesian statistics offer an intellectually acceptable approach to credibility theory. However, credibility mechanisms are difficult to implement in practice, and the merit rating schemes applied by insurance companies are simplified versions of credibility formulas.

One of the commercial simplifications of credibility theory is known as bonus-malus systems. Such systems take the form of a scale comprising a number of levels. The policyholders move inside the scale according to the number of claims they report to the insurance company. To each level of the scale is attached a relativity (that is, a percentage, or relative premium). These relativities are applied to a base premium. Usually, bonus-malus systems may be modelled through a Markov chain, which makes mathematics easy for the actuary. More details on these bonus-malus systems can be found in Norberg (1976) and comprehensively in Lemaire (1995).

France is an exception. The French law imposes to the insurers operating in France a unique bonus-malus system. That bonus-malus system is not based on a scale. Instead the French bonus-malus system uses the concept of increase-decrease coefficient (*coefficient de réduction-majoration* in French, henceforth abbreviated as CRM). More precisely, the French bonus-malus system implies a malus of 25% per claim and a bonus of 5% per claim-free year. So each policyholder is assigned a base premium and this base premium is adapted according to the number of claims reported to the insurer, multiplying the premium by 1.25 each time an accident at fault is reported to the company, and by 0.95 per claim-free year. In case of shared responsibility, the increase is reduced by half (12.5% instead of 25%). Note that these increases are applied to the previous relativity: the first claim causes the premium pass from 100 to 125, the second increases the premium to 156, the third to 195, and so on (all the numbers are rounded down). The penalties are thus convex in the number of claims reported by the driver, ensuring that the more claims are reported, the heavier they are penalized. The highest percentage is 350, and the lowest is 50 (attained after 13 consecutive claim-free years). According to the French special bonus rule, after two consecutive years without a claim, the driver goes back to the initial level 100%. This special bonus rule is particularly generous.

In 1994, the European Union decreed that all its member countries had to drop their mandatory bonus-malus systems, claiming that such systems reduced competition between insurers and were in contradiction with the total rating freedom implemented by the Third Directive. The complete rating freedom became a reality in July 2004. Since that date, Belgium for instance dropped its mandatory bonus-malus system, but all companies operating in Belgium still apply the former uniform system (with minor modifications for the best drivers occupying the lowest levels in the scale, mainly for marketing purposes).

However, the mandatory French system is still in force. Quite surprisingly, the European Court of Justice decided in 2004 that the mandatory bonus-malus systems of France and the Grand Duchy of Luxembourg were not in contradiction to the rating freedom imposed by the European legislation. These two countries were thus allowed to stick to their uniform bonus-malus mechanism.

Bonus-malus systems allow to match individual premium to risk and to increase incentives for road safety, by taking into consideration the past record. They can be justified by asymmetrical information between the insurance company and the policyholders. Indeed, they encourage policyholders to drive carefully (i.e., they counteract moral hazard) and respond to adverse selection in automobile insurance. Adverse selection occurs when the policyholders have a better knowledge of their claim behavior than the insurer does, and take advantage of this additional information about their driving patterns, known to them but unknown to the insurer. Experience rating is a response to adverse selection, by penalizing the more numerous claims of those with more dangerous driving patterns.

It is interesting to confront economists' and actuaries' approaches to experience rating. In the economic literature, discounts and penalties are introduced mainly to counteract inefficiency which arises from moral hazard. In the actuarial literature, the main purpose is to better assess the individual risk so that everyone will pay, in the long run, a premium corresponding to his own claim frequency.

Since the penalty induced by the bonus-malus system is in general independent of the claim amount, a crucial issue for the policyholder is therefore to decide whether it is profitable or not to report small claims (in order to avoid an increase in premium). Cheap claims are likely to be defrayed by the policyholders themselves, and not to be reported to the company. This phenomenon, known as the hunger for bonus, limits claim handling costs since small claims are not reported to the insurer (decreasing the administrative burden).

In this paper, we show that the framework of credibility theory can be used to analyze the French bonus-malus system. Specifically, the greatest accuracy credibility approach is adapted to fit the CRM coefficients: the actuary resorts to a quadratic loss function but the shape of the credibility predictor is constrained ex ante to the form imposed by the French law. Let us mention that the approach developed in this paper is not the only possible to deal with CRM coefficients. It has been shown in KELLE (2000) that the French bonus-malus system corresponds to a scale comprising several hundreds of levels (530 levels, precisely), that can be analyzed in the Markovian setting of NORBERG (1976). The large amount of states needed is due to the malus reduction in case of claims with shared responsability, forcing the author to consider the pair (number of claims with whole responsibility, number of claims with partial liability) to make the computation. The form of the transition matrix is somehwat intricate and we believe that the alternative developed in this paper offers an appropriate treatment of the CRM's.

Let us now detail the contents of this paper. In Section 2, we model the CRM's and we compute the parameters involved in the French bonus-malus system. We also examine whether the bonus-malus system is financially balanced or not. Some numerical applications illustrate the methodological results. Section 3 discusses a special rule associated to the French bonus-malus system: claims for which the policyholder is only partially liable entail a reduced penalty. The impact of this reduction is evaluated, and numerical illustrations are discussed. The final Section 4 concludes.

# 2 Fitting the French bonus-malus system

## 2.1 Modelling claim frequencies

Let us pick at random a policyholder from the portfolio. We denote as $N_t$ the number of claims reported by this policyholder in period $t$. We assume that $N_t$ is Poisson distributed with parameter $\lambda\Theta$ where $\Theta$ is a random effect accounting for the heterogeneity present in the portfolio. By assumption, $\Theta$ is a positive random variable such that $\mathbb{E}[\Theta] = 1$, so that $\mathbb{E}[N] = \lambda$, which represents the annual mean frequency in the portfolio (or in the risk class in case of a segmented tariff). The conditional probability mass function of $N_t$ is given by

$$\Pr[N_t = k|\Theta = \theta] = e^{-\lambda\theta}\frac{(\lambda\theta)^k}{k!} \quad , \quad k = 0, 1, 2, \ldots$$

The heterogeneity present in the portfolio is described by a structure function $u$. Formally, $u$ is the probability density function of $\Theta$. Therefore, the unconditional probability mass function of $N_t$ is given by

$$\Pr[N_t = k] = \int_0^{+\infty} \Pr[N_t = k|\Theta = \theta]u(\theta)d\theta.$$

Furthermore, the random variables $N_1, N_2, N_3, \ldots$ are assumed to be independent and identically distributed given the risk proneness $\Theta$ of the policyholder. Since $\Theta$ is unknown to the insurer, this induces serial dependence among the $N_t$'s.

## 2.2 CRM coefficients

We will assume that the CRM coefficients only depend on the observed number of reported claims and not on their severity. Therefore the base premium is simply $\lambda$ multiplied by a constant (essentially the expected cost of a claim).

Let $\epsilon_t$ be the "reduction" coefficient and $\delta_t$ be the "majoration" coefficient applying to a policyholder who has been covered for $t$ years. The CRM coefficient for years 1 to $t$ then writes

$$r_{\delta_t,\epsilon_t}(N_\bullet, I_\bullet, t) = (1 + \delta_t)^{N_\bullet}(1 - \epsilon_t)^{I_\bullet}$$

with

$$N_\bullet = \sum_{j=1}^{t} N_j \text{ and } I_\bullet = \sum_{j=1}^{t} I_j, \tag{2.1}$$

where $I_j$ is defined as

$$I_j = \begin{cases} 1 \text{ if } N_j = 0 \\ 0 \text{ if } N_j \geq 1. \end{cases}$$

In words, $N_\bullet$ is the total number of claims reported by the policyholder during the period $(0, t)$ and $I_\bullet$ is the number of years without any claim reported to the company. Note that the CRM coefficients depend on $t$, so that the penalties and discounts may change for every $(0, t)$ period.

We will use Norberg (1976)'s criterion in order to obtain the parameters $\epsilon_t$ and $\delta_t$, i.e. we minimize the expected squared difference between the "true" relative premium $\Theta$ and the relative premium $r_{\delta_t,\epsilon_t}$ applicable to the policyholder according to the French-type bonus-malus system. More specifically, for a policyholder observed during $t$ years, and having caused $N_1, N_2, \ldots, N_t$ claims, we aim to determine $\epsilon_t$ and $\delta_t$ so to minimize the objective function

$$\Psi_t(\delta, \epsilon) = \mathbb{E}[(\Theta - r_{\delta,\epsilon}(N_\bullet, I_\bullet, t))]^2$$

with respect to the parameters $\epsilon$ and $\delta$. We therefore have to solve the first order conditions

$$\frac{\partial}{\partial \delta}\Psi_t(\epsilon, \delta) = 0 \quad \text{and} \quad \frac{\partial}{\partial \epsilon}\Psi_t(\epsilon, \delta) = 0$$

which rewrites

$$\begin{cases} \mathbb{E}\left[\Theta N_\bullet (1+\delta)^{N_\bullet - 1}(1-\epsilon)^{I_\bullet}\right] = \mathbb{E}\left[N_\bullet (1+\delta)^{2N_\bullet - 1}(1-\epsilon)^{2I_\bullet}\right] \\ \\ \mathbb{E}\left[\Theta (N_\bullet)(1+\delta)^{N_\bullet}(1-\epsilon)^{I_\bullet - 1}\right] = \mathbb{E}\left[N_\bullet (1+\delta)^{2N_\bullet}(1-\epsilon)^{2I_\bullet - 1}\right]. \end{cases} \tag{2.2}$$

## 2.3 Computation of the CRM's at time $t$

Let us define the conditional probability generating function of the random couple $(N_\bullet, I_\bullet)$ given $\Theta = \theta$ as

$$\phi(\xi_1, \xi_2|\theta) = \mathbb{E}[\xi_1^{N_\bullet}\xi_2^{I_\bullet}|\Theta = \theta].$$

The conditional independence assumption of $N_1, N_2, \ldots, N_t$ allows us to write

$$\begin{aligned} \phi(\xi_1, \xi_2|\theta) &= \prod_{j=1}^{t} \mathbb{E}\left[\xi_1^{N_j}\xi_2^{I_j}|\Theta = \theta\right] \\ &= \prod_{j=1}^{t}\left(e^{-\lambda\theta}(\xi_2 - 1) + e^{-\lambda\theta(1-\xi_1)}\right) \\ &= \left(e^{-\lambda\theta}(\xi_2 - 1) + e^{-\lambda\theta(1-\xi_1)}\right)^t. \end{aligned}$$

We can then rewrite the system (2.2) as

$$\begin{cases} 2\mathbb{E}[\Theta\phi^{(1,0)}(1+\delta, 1-\epsilon|\Theta)] = \mathbb{E}[\phi_2^{(1,0)}(1+\delta, 1-\epsilon|\Theta)] \\ \\ 2\mathbb{E}[\Theta\phi^{(0,1)}(1+\delta, 1-\epsilon|\Theta)] = \mathbb{E}[\phi_2^{(0,1)}(1+\delta, 1-\epsilon|\Theta)] \end{cases}$$

where

$$\begin{aligned}
\phi^{(x,y)}(a,b|\theta) &= \frac{\partial^x\partial^y}{\partial s^x \partial t^y}\phi(s,t|\theta)\Big|_{s=a,t=b} \\
\phi_2^{(x,y)}(a,b|\theta) &= \frac{\partial^x\partial^y}{\partial s^x \partial t^y}\phi(s^2,t^2|\theta)\Big|_{s=a,t=b}
\end{aligned}$$

for $x, y \in \{0,1\}$.

We can rewrite the first order conditions as

$$\begin{aligned}
&\int_0^\infty \theta^2\Big(e^{\lambda\theta\delta} - \epsilon e^{-\lambda\theta}\Big)^{t-1} e^{\lambda\theta\delta} u(\theta)d\theta \\
&= (1+\delta)\int_0^\infty \theta\Big(e^{\lambda\theta(2\delta+\delta^2)} + e^{-\lambda\theta}(\epsilon^2 - 2\epsilon)\Big)^{t-1} e^{\lambda\theta(2\delta+\delta^2)} u(\theta)d\theta \\
&\int_0^\infty \theta\Big(e^{\lambda\theta\delta} - \epsilon e^{-\lambda\theta}\Big)^{t-1} e^{-\lambda\theta} u(\theta)d\theta \\
&= (1-\epsilon)\int_0^\infty \Big(e^{\lambda\theta(2\delta+\delta^2)} + e^{-\lambda\theta}(\epsilon^2 - 2\epsilon)\Big)^{t-1} e^{-\lambda\theta} u(\theta)d\theta.
\end{aligned}$$

These equations do not possess a closed form solution as is the case for Markovian systems (see, e.g., NORBERG (1976)). Nevertheless, they can be solved numerically using on the one hand a numerical integration algorithm and on the other hand either an algorithm allowing to numerically solve a system of non linear equations or an optimisation algorithm, depending on what type of procedure is available. In the numerical illustrations proposed in this paper, we have used an optimisation algorithm from the SAS/IML package, trying to minimize the sum of the squared differences between the left-hand side and the right-hand side of the two equations displayed above. The main structure of the SAS/IML optimization algorithm is provided in the appendix to this paper.

## 2.4 Global CRM

Note that we have obtained so far a numerical solution for each $t$: minimizing $\Psi_t(\delta,\epsilon)$ with respect to $\epsilon$ and $\delta$ gives the optimal solution $(\epsilon_t,\delta_t)$ for the period $(0,t)$. However we want to obtain a unique set of CRM coefficients. These may be obtained in the spirit of BORGAN ET AL. (1981). To this end, let us introduce the age structure of the portfolio. Specifically, we denote as $A$ the number of years the driver is covered by the company, and as $N_1, N_2, \ldots, N_A$ the annual numbers of claims reported by this policyholder. Note that $A$ is a random variable since we work with a policyholder picked at random from the portfolio. The idea is then to determine $\epsilon$ and $\delta$ so to minimize $\mathbb{E}[\Psi_A(\epsilon,\delta)]$. The objective function then becomes

$$\Psi(\epsilon,\delta) = \mathbb{E}[\Psi_A(\epsilon,\delta)] = \sum_{t=1}^{\infty}\omega_t\Psi_t(\epsilon,\delta) \text{ where } \omega_t = \Pr[A=t],$$

to be minimized with respect to the parameters $\epsilon$ and $\delta$.

Some algebra immediately leads to the following system to solve

$$\begin{aligned}
&\sum_{t=1}^{\infty}\omega_t t\int_0^{\infty}\theta^2\Big(e^{\lambda\theta\delta}-\epsilon e^{-\lambda\theta}\Big)^{t-1}e^{\lambda\theta\delta}u(\theta)d\theta\\
=\;&(1+\delta)\sum_{t=1}^{\infty}\omega_t t\int_0^{\infty}\theta\Big(e^{\lambda\theta(2\delta+\delta^2)}+e^{-\lambda\theta}(\epsilon^2-2\epsilon)\Big)^{t-1}e^{\lambda\theta(2\delta+\delta^2)}u(\theta)d\theta\\
&\sum_{t=1}^{\infty}\omega_t t\int_0^{\infty}\theta\Big(e^{\lambda\theta\delta}-\epsilon e^{-\lambda\theta}\Big)^{t-1}e^{-\lambda\theta}u(\theta)d\theta\\
=\;&(1-\epsilon)\sum_{t=1}^{\infty}\omega_t t\int_0^{\infty}\Big(e^{\lambda\theta(2\delta+\delta^2)}+e^{-\lambda\theta}(\epsilon^2-2\epsilon)\Big)^{t-1}e^{-\lambda\theta}u(\theta)d\theta.
\end{aligned}$$

Again, this system does not admit any closed-form solution, but can be solved numerically. The main structure of the SAS/IML optimization algorithm is provided in the appendix to this paper.

*Remark* 2.1. Note that here, we have made an averaging with respect to the age structure of the portfolio. In case the portfolio is partitioned in a series of risk classes, an average with respect to the composition of the portfolio (in terms of classification variables) could also be performed. If some explanatory variables are correlated with $A$, care must be taken in the second averaging.

## 2.5 Analysis of the financial equilibrium of the French bonus-malus system

An interesting property of the relativities associated to Markovian bonus-malus systems and obtained through the Norberg's least-squares criterion is that they make the bonus-malus system financially balanced, i.e. the premium income of the insurer does not increase nor decrease over time (on average). In this section, we would like to check whether or not the French-type bonus-malus system enjoys this property.

More precisely, once $\epsilon_t$ and $\delta_t$ have been obtained, we would like to verify whether the equality

$$\mathbb{E}[r_{\epsilon_t,\delta_t}(N_\bullet, I_\bullet, t)] = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\Pr[N_\bullet = x, I_\bullet = y]\, r_{\epsilon_t,\delta_t}(x,y,t) = 1$$

holds true, where $N_\bullet$ and $I_\bullet$ are as defined in (2.1). The computation of $\mathbb{E}[r_{\epsilon_t,\delta_t}(N_\bullet, I_\bullet, t)]$ requires the knowledge of the joint distribution of the random couple $(N_\bullet, I_\bullet)$.

Let us denote as $f(x,y|\theta) = \Pr[N_1 = x, I_1 = y|\Theta = \theta]$ the joint discrete mass function of the random couple $(N_1, I_1)$, conditional on $\Theta = \theta$, and as $f^{\star(t)}(x,y|\theta) = \Pr[N_\bullet = x, I_\bullet = y|\Theta = \theta]$ the joint discrete mass function of the random couple $(N_\bullet, I_\bullet)$ defined in (2.1), conditional on $\Theta = \theta$. We then have the following result.

**Theorem 2.2.** *For fixed $\theta$, the following recursive formulas*

$$\begin{array}{rcl}
g^{\star(t)}(x,y|\theta) & = & f^{\star(t)}(x,t-y|\theta) \text{ for } 0 \le y \le t \text{ and } x > 0 \\
g^{\star(t)}(0,0|\theta) & = & e^{-\lambda\theta t} \\
f(x,0|\theta) & = & e^{-\lambda\theta}\dfrac{(\lambda\theta)^x}{x!} \text{ for } x > 0 \\
f(0,1|\theta) & = & e^{-\lambda\theta} \\
g^{\star(t)}(x,y|\theta) & = & e^{\lambda\theta}\displaystyle\sum_{u=1}^{x}\left(\frac{t+1}{y}-1\right) g^{\star(t)}(x-u,y-1|\theta)g(u,1|\theta) \text{ for } y \ge 1 \text{ and } x \ge 1,
\end{array}$$

*hold true, with the convention that the functions take the value 0 where they are not defined.*

*Proof.* It is trivial that for $t = 1$ we have

$$\begin{array}{rcl}
f(x,0|\theta) & = & e^{-\lambda\theta}\dfrac{(\lambda\theta)^x}{x!} \quad , \quad x > 0 \\
f(0,1|\theta) & = & e^{-\lambda\theta}.
\end{array}$$

As $f^{\star(t)}$ is the $t$-fold convolution of a lattice random vector, we are in a position to apply the bivariate extension of De Pril's algorithm (that can be found, e.g., in Sundt (1999)). As that algorithm needs an atom at the origin, we define an auxilliary density function $g^{\star(t)}(x,y|\theta) = f^{\star(t)}(x,t-y|\theta)$. We find

$$\begin{array}{rcl}
g^{\star(t)}(0,0|\theta) & = & e^{-\lambda\theta t} \\
g^{\star(t)}(x,y|\theta) & = & e^{\lambda\theta}\left(\displaystyle\sum_{u=0}^{x}\left(\frac{t+1}{x}u-1\right) g^{\star(t)}(x-u,y-1|\theta)g(u,1|\theta)\right. \\
 & & \left. +\displaystyle\sum_{u=1}^{x}\left(\frac{t+1}{x}u-1\right) g^{\star(t)}(x-u,y|\theta)g(u,0|\theta)\right) \quad , \quad x \ge 1 \\
g^{\star(t)}(x,y|\theta) & = & e^{\lambda\theta}\left(\displaystyle\sum_{u=0}^{x}\left(\frac{t+1}{y}-1\right) g^{\star(t)}(x-u,y-1|\theta)g(u,1|\theta)\right. \\
 & & \left. +\displaystyle\sum_{u=1}^{x}(-1)g^{\star(t)}(x-u,y|\theta)g(u,0|\theta)\right) \quad , \quad y \ge 1.
\end{array}$$

Because $g(0,1|\theta) = 0$ and $g(x,0|\theta) = 0$ for $x > 0$, we obtain

$$\begin{array}{rcl}
g^{\star(t)}(0,0|\theta) & = & e^{-\lambda\theta t} \\
g^{\star(t)}(x,y|\theta) & = & e^{\lambda\theta}\displaystyle\sum_{u=1}^{x}\left(\frac{t+1}{x}u-1\right) g^{\star(t)}(x-u,y-1|\theta)g(u,1|\theta), \quad x \ge 1 \\
g^{\star(t)}(x,y|\theta) & = & e^{\lambda\theta}\displaystyle\sum_{u=1}^{x}\left(\frac{t+1}{y}-1\right) g^{\star(t)}(x-u,y-1|\theta)g(u,1|\theta), \quad y \ge 1.
\end{array}$$

Because $g^{\star(t)}(x,0|\theta) = 0$ for $x > 0$, only the second recursive formula has to be used. This formula is stable because $\frac{t+1}{y} - 1 > 0$. □

In order to obtain $f^{\star(t)}(x,y)=\Pr[N_\bullet=x,I_\bullet=y]$, it suffices to integrate the conditional mass function $f^{\star(t)}(x,y|\theta)$ with respect to the structure function $u$, that is,

$$f^{\star(t)}(x,y)=\int_0^\infty f^{\star(t)}(x,y|\theta)u(\theta)d\theta \quad , \quad x>0 \quad , \quad 0\le y\le t.$$

These quantities can then be used to evaluate $\mathbb{E}[r_{\epsilon_t,\delta_t}(N_\bullet,I_\bullet,t)]$.

## 2.6 Numerical application

We will assume that $\Theta$ is Gamma distributed with probability density function

$$u(\theta)=\frac{1}{\Gamma(a)}a^a\theta^{a-1}e^{-a\theta} \quad , \quad \theta\ge 0.$$

The parameters $a$ and $\lambda$ are estimated on the basis of a Belgian motor insurance portfolio. This gives $\widehat{\lambda}=0.1474$ and $\widehat{a}=0.8888$.

Table 2.1 displays for different values of $t$, the coefficients $\delta_t$ and $\epsilon_t$ obtained by solving the system given in Section 2.3. We observe a dramatic decrease of the values of $\epsilon_t$ and $\delta_t$ over time. The last column of the table allows us to verify the financial equilibrium of the system. The total premium income first decreases to 97.61% and then increases to 107.29% after 30 years. The discount per claim-free year decreases from 14.23% to about 1%. Similarly, the penalty induced by each reported claim decreases from 61.45% to 6.09%. The a posteriori corrections are therefore considerably softened with time.

| $t$ | $\delta_t$ | $\epsilon_t$ | Financial equilibrium |
|---|---|---|---|
| 1 | 0.6145 | 0.1423 | 0.9761 |
| 2 | 0.4595 | 0.0955 | 0.9985 |
| 3 | 0.3690 | 0.0727 | 1.0001 |
| 4 | 0.3092 | 0.0589 | 1.0099 |
| 10 | 0.1585 | 0.0279 | 1.0431 |
| 20 | 0.0880 | 0.0149 | 1.0638 |
| 30 | 0.0609 | 0.0102 | 1.0729 |

Table 2.1: Parameters $\delta_t$ and $\epsilon_t$ and financial equilibrium for different values of $t$.

The decrease of $\epsilon_t$ and $\delta_t$ with time $t$ that is apparent from Table 2.1 can be explained as follows. The aim is that $r_{\epsilon_t,\delta_t}$ be as close as possible to the unknown risk parameter $\Theta$. Since $\Theta$ does not depend on $t$ whereas $N_\bullet$ and $I_\bullet$ are almost surely non-decreasing with $t$, the optimal parameters $\epsilon_t$ and $\delta_t$ must decrease to compensate for the increase in $N_\bullet$ and $I_\bullet$. This is why averaging over time is needed.

Table 2.2 gives the CRM coefficient $r_{\epsilon_t,\delta_t}(x,y)=(1+\delta_t)^x(1-\epsilon_t)^y$ for different periods of length $t$ and for different values of the total number of claims $x$. The index $t.y$ means that we have $y$ claims-free years during the period $(0,t)$. For the sake of comparison, Table 2.3 gives the CRM coefficients obtained from classical Bayesian credibility. In this case, the a priori annual expected claim frequency is multiplied by $(a+x)/(a+t)$. We observe some large discrepancies between the values listed in Tables 2.2 and 2.3.

| $t.y\backslash x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 1 | 86% | 161% | 261% | 421% | 680% | 1097% | 1771% |
| $2.\geq 1$ | 82% | 132% | 193% | 281% | 410% | 599% | 874% |
| 2.0 | | | 213% | 311% | 454% | 662% | 967% |
| $3.\geq 2$ | 80% | 118% | 161% | 221% | 302% | 414% | 566% |
| 3.1 | | | 174% | 238% | 326% | 446% | 610% |
| 3.0 | | | | 257% | 351% | 481% | 658% |

Table 2.2: CRM coefficients $r_{\epsilon_t,\delta_t}(x,y)$ for different values of $t$, $x$ and $y$.

| $t\backslash x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 1 | 86% | 182% | 279% | 375% | 472% | 568% | 665% |
| 2 | 75% | 160% | 244% | 329% | 413% | 498% | 582% |
| 3 | 67% | 142% | 217% | 292% | 367% | 442% | 518% |

Table 2.3: Premium update coefficients derived from Bayesian credibility in the Poisson-Gamma model.

We see from Tables 2.2 and 2.3 that the discounts awarded to the policyholders who did not report any claim (column $x = 0$ in Tables 2.2 and 2.3) are larger with Bayesian credibility than with CRM coefficients. From the approximate financial stability evidenced in Table 2.1, the penalties induced by CRM coefficients must therefore be softer compared to Bayesian credibility. For instance, policyholders who reported a single claim (column $x = 1$ in Tables 2.2 and 2.3) have premium surcharge ranging from 118 to 161% with CRM coefficients, and ranging from 142 to 182% with Bayesian credibility. However, policyholders reporting many claims are more heavily penalized with CRM coefficients than with Bayesian credibility. This comes from the convex behavior of the CRM coefficients, whereas Bayesian credibility corrections are linear in the past number of claims.

To obtain unique values for the CRM coefficients, we have to decide about an age structure of the policies comprised in the portfolio. Here, we take the following hypothetical distribution of the portfolio :

$$\begin{array}{rcl}
\omega_1 & = & 10\% \\
\omega_5 & = & 20\% \\
\omega_{12} & = & 30\% \\
\omega_{20} & = & 20\% \\
\omega_{25} & = & 10\% \\
\omega_{30} & = & 10\% \\
\omega_t & = & 0 \text{ for all other } t.
\end{array}$$

The minimization of $\mathbb{E}[\Psi_A(\epsilon,\delta)] = \sum_{t=1}^{\infty} \omega_t \Psi_t(\epsilon,\delta)$ with respect to $\epsilon$ and $\delta$ then gives the optimal solutions $\delta = 0.0710$ and $\epsilon = 0.0133$. The financial equilibrium is achieved, as the total premium income tends to 104.29% of the initial one. When working with a weighted average of the $\Psi_t(\epsilon,\delta)$'s, the values associated to large $t$'s play the prominent role, resulting in values for $\delta$ and $\epsilon$ similar to those obtained for $t > 20$ in Table 2.1.

With optimal CRM coefficients, the discount for claim-free policyholders is rather modest (1.33% per claim-free year), but the penalty in case of a claim is also moderate (7.1%). The large differences compared with the official values of today's bonus-malus system in France (5% of discount per claim-free year, and 25% increase per claim) can be explained by the fact that all the penalties are suppressed after two claim-free years according to the terms of the French law, which is particularly generous.

We have also tested two different sets of $\omega_t$'s, to study the influence of the age structure of the portfolio on the optimal CRM coefficients. With the age structure of an "old" portfolio, that is,

$$\begin{aligned}
\omega_1 &= 10\% \\
\omega_5 &= 10\% \\
\omega_{12} &= 10\% \\
\omega_{20} &= 20\% \\
\omega_{25} &= 20\% \\
\omega_{30} &= 30\% \\
\omega_t &= 0 \text{ for all other } t,
\end{aligned}$$

the minimization of $\mathbb{E}[\Psi_A(\epsilon, \delta)]$ gives $\delta = 0.0658$, $\epsilon = 0.0115$. With the age structure of a "young" portfolio, that is,

$$\begin{aligned}
\omega_1 &= 30\% \\
\omega_5 &= 20\% \\
\omega_{12} &= 20\% \\
\omega_{20} &= 10\% \\
\omega_{25} &= 10\% \\
\omega_{30} &= 10\% \\
\omega_t &= 0 \text{ for all other } t,
\end{aligned}$$

the minimization of $\mathbb{E}[\Psi_A(\epsilon, \delta)]$ gives $\delta = 0.0694$, $\epsilon = 0.0132$. The influence of the age structure on the optimal CRM coefficients is thus rather moderate.

# 3 Partial liability

## 3.1 Reduced penalty and modelling claim frequencies

The French bonus-malus system possesses many particular rules. This section is devoted to the study of one of them. Specifically, according to the terms of the French law, if the policyholder is partially liable for the claim then the premium is multiplied by 1.125 instead of 1.25. To take such a rule into account, we have to model the random couple $(N_{1t}, N_{2t})$ where $N_{1t}$ counts the number of full liability claims filed during year $t$ and $N_{2t}$ counts the number of partial liability claims filed during the same year. Clearly, $N_{1t} + N_{2t}$ is the total number of claims $N_t$ used in the preceding section.

Let $q$ be the probability that the policyholder is only partially liable for the claim he files. Further, let us assume a Bernoulli scheme for the claim types. This ensures that, conditionally on $\Theta$, $N_{1t}$ and $N_{2t}$ are independent and both conform to the Poisson distribution.

Specifically, we have now

$$\begin{aligned}
\Pr[N_{1t}=k|\Theta=\theta] &= e^{-\lambda\theta(1-q)}\frac{(\lambda\theta(1-q))^k}{k!} \quad , \quad k=0,1,2,\ldots, \\
\Pr[N_{2t}=k|\Theta=\theta] &= e^{-\lambda\theta q}\frac{(\lambda\theta q)^k}{k!} \quad , \quad k=0,1,2,\ldots
\end{aligned}$$

The random variables $N_{1t}$ and $N_{2t}$ are obviously dependent if the risk proneness $\Theta$ is unknown. The joint probability mass for the random couple $(N_{1t}, N_{2t})$ is given by

$$\Pr[N_{1t}=k_1, N_{2t}=k_2] = \int_0^{+\infty} \Pr[N_{1t}=k_1|\Theta=\theta]\Pr[N_{2t}=k_2|\Theta=\theta]u(\theta)d\theta.$$

This is a mixed bivariate Poisson model in the spirit of Walhin & Paris (2001).

## 3.2 Computations of the CRM's at time $t$

Let us consider a policyholder covered for $t$ years. In addition to the parameter $\delta_t$ giving the magnitude of the penalty in case of a full-liability claim, we introduce the new parameter $\gamma_t$ giving the reduced penalty in case of a partial liability claim. Now the CRM coefficient for the time period $(0,t)$ writes

$$r_{\delta_t,\gamma_t,\epsilon_t}(N_{1\bullet}, N_{2\bullet}, I_{12\bullet}, t) = (1+\delta_t)^{N_{1\bullet}}(1+\gamma_t)^{N_{2\bullet}}(1-\epsilon_t)^{I_{12\bullet}}$$

with

$$\begin{aligned}
N_{1\bullet} &= \sum_{j=1}^{t} N_{1j} \\
N_{2\bullet} &= \sum_{j=1}^{t} N_{2j} \\
I_{12\bullet} &= \sum_{j=1}^{t} I_j \text{ with } I_j = \begin{cases} 1 \text{ if } N_{1j}=N_{2j}=0 \\ 0 \text{ else.} \end{cases}
\end{aligned}$$

We will assume that $\delta_t = \alpha\gamma_t$ with $\alpha$ fixed by the actuary. The value of $\alpha$ describes the way a claim with full liability is penalized, compared to a claim with partial liability. Then the CRM coefficient re-writes

$$r_{\gamma_t,\epsilon_t}(N_{1\bullet}, N_{2\bullet}, I_{12\bullet}, t) = (1+\alpha\gamma_t)^{N_{1\bullet}}(1+\gamma_t)^{N_{2\bullet}}(1-\epsilon_t)^{I_{12\bullet}}$$

In order to obtain $\epsilon_t$ and $\gamma_t$ we have now to minimize the objective function

$$\Psi_t(\gamma,\epsilon) = \mathbb{E}\big[\big(\Theta - r_{\gamma,\epsilon}(N_{1\bullet}, N_{2\bullet}, I_{12\bullet}, t)\big)^2\big]$$

with respect to the parameters $\gamma$ and $\epsilon$.

Let us define $\phi(\xi_1, \xi_2, \xi_3|\theta) = \mathbb{E}[\xi_1^{N_{1\bullet}} \xi_2^{N_{2\bullet}} \xi_3^{I_{12\bullet}} |\Theta = \theta]$ the conditional probability generating function of the random vector $(N_{1\bullet}, N_{2\bullet}, I_{12\bullet})$ given $\Theta = \theta$. We clearly have that

$$\phi(\xi_1, \xi_2, \xi_3|\theta) \;=\; \left(e^{-\lambda\theta}(\xi_3 - 1) + e^{\lambda\theta\left((1-q)\xi_1 + q\xi_2 - 1\right)}\right)^t.$$

It can be verified that the first order conditions are as follows:

$$\begin{aligned} & 2\mathbb{E}_\Theta[\Theta\alpha\phi^{(1,0,0)}(1+\alpha\gamma_t, 1+\gamma_t, 1-\epsilon_t|\Theta) + \Theta\phi^{(0,1,0)}(1+\alpha\gamma_t, 1+\gamma_t, 1-\epsilon_t|\Theta)] \\ = \;& \mathbb{E}_\Theta[\alpha\phi_2^{(1,0,0)}(1+\alpha\gamma_t, 1+\gamma_t, 1-\epsilon_t|\Theta) + \phi_2^{(0,1,0)}(1+\alpha\gamma_t, 1+\gamma_t, 1-\epsilon_t|\Theta)] \end{aligned}$$

and

$$2\mathbb{E}_\Theta[\Theta\phi^{(0,0,1)}(1+\alpha\gamma_t, 1+\gamma_t, 1-\epsilon_t|\Theta)] \;=\; \mathbb{E}_\Theta[\phi_2^{(0,0,1)}(1+\alpha\gamma_t, 1+\gamma_t, 1-\epsilon_t|\Theta)]$$

where

$$\begin{aligned} \phi^{(x,y,z)}(a,b,c|\theta) &= \frac{\partial^x\partial^y\partial^z}{\partial s^x \partial t^y \partial u^z}\phi(s,t,u|\theta)|_{s=a,t=b,u=c} \\ \phi_2^{(x,y,z)}(a,b,c|\theta) &= \frac{\partial^x\partial^y\partial^z}{\partial s^x \partial t^y \partial u^z}\phi(s^2,t^2,u^2|\theta)|_{s=a,t=b,u=c} \end{aligned}$$

for $x, y, z \in \{0, 1\}$. Again, numerical procedures are needed to find the solution of this optimization problem.

## 3.3 Financial equilibrium

Analyzing the financial equilibrium of the system now amounts to check whether the equality

$$\mathbb{E}[r_{\gamma_t,\epsilon_t}(N_{1\bullet}, N_{2\bullet}, I_{12\bullet}, t)] = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\sum_{z=0}^{\infty}\Pr[N_{1\bullet} = x, N_{2\bullet} = y, N_{12\bullet} = z] r_{\gamma_t,\epsilon_t}(x,y,z,t) = 1$$

holds true with the optimal values $\gamma_t$ and $\epsilon_t$.

The joint distribution of the random vector $(N_{1\bullet}, N_{2\bullet}, I_{12\bullet})$ is given in the following theorem that extends Theorem 2.2 in the present setting.

**Theorem 3.1.** *For fixed $\theta$, the following recursive formulas*

$$\begin{aligned} g^{\star(t)}(x,y,z|\theta) &= f^{\star(t)}(x,y,t-z|\theta), \text{ for } 0 \le z \le t, \;\; x,y \ge 0 \text{ and } x+y > 0 \\ g^{\star(t)}(0,0,0|\theta) &= e^{-\lambda\theta t} \\ f(x,y,0|\theta) &= e^{-\lambda\theta}\frac{(\lambda\theta)^{x+y}(1-q)^x q^y}{x!y!} \text{ for } x,y \ge 0 \text{ and } x+y > 0, \\ f(0,0,1|\theta) &= e^{-\lambda\theta} \\ g^{\star(t)}(x,y,z|\theta) &= e^{\lambda\theta}\sum_{u=0}^{x}\sum_{v=0}^{y}\left(\frac{t+1}{z} - 1\right) g^{\star(t)}(x-u, y-v, z-1|\theta) g(u,v,1|\theta), \\ & \qquad \text{for } 1 \le z \le t, \;\; x,y \ge 0 \text{ and } x+y > z-1 \end{aligned}$$

*hold true with the convention that the defined functions take the value 0 where they are not defined.*

### 3.4 Numerical illustrations

To illustrate this special case, we use the same parameters as in Section 2.6. We assume that 20% of the claims concerns partial liability, that is $q = 0.2$. We numerically solve the system of two equations for different values of $\alpha$. Table 3.1 gives the results. As before, $\gamma_t$ and $\epsilon_t$ decrease with $t$ and an averaging is needed to get a unique set of parameters

| | $\alpha = 1.5$ | | | $\alpha = 2.0$ | | | $\alpha = 2.5$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $t$ | $\gamma_t$ | $\epsilon_t$ | Fin. equilibrium | $\gamma_t$ | $\epsilon_t$ | Fin. equilibrium | $\gamma_t$ | $\epsilon_t$ | Fin. equilibrium |
| 1 | 0.4336 | 0.1423 | 0.9747 | 0.3316 | 0.1423 | 0.9729 | 0.2674 | 0.1423 | 0.9714 |
| 2 | 0.3253 | 0.0954 | 0.9870 | 0.2498 | 0.0953 | 0.9850 | 0.2022 | 0.0953 | 0.9832 |
| 3 | 0.2616 | 0.0727 | 0.9984 | 0.2014 | 0.0726 | 0.9965 | 0.1633 | 0.0726 | 0.9946 |
| 4 | 0.2194 | 0.0589 | 1.0083 | 0.1692 | 0.0589 | 1.0062 | 0.1374 | 0.0588 | 1.0047 |
| 10 | 0.1128 | 0.0279 | 1.0419 | 0.0873 | 0.0279 | 1.0402 | 0.0711 | 0.0279 | 1.0386 |
| 20 | 0.0627 | 0.0149 | 1.0634 | 0.0486 | 0.0149 | 1.0624 | 0.0397 | 0.0149 | 1.0617 |
| 30 | 0.0435 | 0.0102 | 1.0725 | 0.0337 | 0.0102 | 1.0707 | 0.0276 | 0.0102 | 1.0712 |

Table 3.1: Parameters $\gamma_t$ and $\epsilon_t$ and financial equilibrium for different values of $t$ and $\alpha$.

The total income of the company is not much influenced by the value of $\alpha$, and is quite close to the values listed in Table 2.1.

To obtain unique values for the CRM coefficients, we choose the first age distribution of Section 2.6. The minimization of

$$\mathbb{E}[\Psi_A(\gamma, \epsilon)] = \sum_{t=1}^{\infty} \omega_t \Psi_t(\gamma, \epsilon)$$

then gives the values displayed in Table 3.2. The same comments apply to this case. Specifically, the $\Psi_t(\gamma, \epsilon)$'s with large values of $t$ play the prominent role, giving optimal CRM coefficients close to the values obtained with $t > 20$ in Table 2.1.

| $\alpha = 1.5$ | | | $\alpha = 2.0$ | | | $\alpha = 2.5$ | | |
|---|---|---|---|---|---|---|---|---|
| $\gamma$ | $\epsilon$ | Fin. equilibrium | $\gamma$ | $\epsilon$ | Fin. equilibrium | $\gamma$ | $\epsilon$ | Fin. equilibrium |
| 0.0507 | 0.0133 | 1.0419 | 0.0393 | 0.0133 | 1.0403 | 0.0321 | 0.0133 | 1.0393 |

Table 3.2: Parameters $\gamma$ and $\epsilon$ and financial equilibrium for different values $\alpha$.

The values of the optimal CRM coefficients displayed in Table 3.2 are again much smaller than those implemented by the French law. As before, this is due to the special bonus rule of the French system (after two consecutive years without claim, the driver goes back to the initial level of 100%).

# 4 Discussion

To sum up, we have analyzed bonus-malus systems similar to the French one: policyholders do not move inside a scale but are subject to CRM coefficients. The parameters are determined by a least-squares (or maximum accuracy) criterion for a fixed time horizon, and then averaged with respect to the age structure of the portfolio (through weighted least-squares) to produce a unique set of CRM coefficients. The financial balance is approximately achieved (the total premium income slightly increases when the bonus-malus system is in force for a few years) and the optimal discounts and penalties are much smaller than the ones applied in France. This is certainly due to the special bonus rule, that has been disregarded in the present analysis.

In the Poisson-Gamma (or Negative Binomial) case, the penalties corresponding to CRM coefficients are convex functions of the number of claims reported in the past, whereas corrections induced by credibility mechanisms are linear in this number. Compared with credibility, the bonus-malus system grants less discounts, penalizes policyholders reporting a single claim to a lesser extent but induces more severe premium corrections for those reporting at least two claims.

An extension to several types of claims is proposed. Specifically, as in the French case, claims with partial liability are distinguished from claims with full liability. In the former case, the penalty is decreased.

In this paper, we did not consider all the characteristics of the bonus-malus system in force in France. We have disregarded the special bonus rule (which suppresses all the penalties after two claim-free years). The French law imposes other special rules to insurance companies. For instance, the French bonus-malus system is such that drivers never pay more than 350% of the base premium nor less than 50% of the base premium. Therefore the minimization process has to be carried with an adapted CRM coefficient of the form $r^*_{\epsilon,\delta} = \max(0.5, \min(3.5, r_{\epsilon,\delta}))$.

Several simplifying assumptions can be considered to ease the numerical computations. For instance, we could work with binary annual claim numbers: either the policyholder does not report any claim or he reports a single claim to the company. Such an assumption replacing $N_t$ by $\min\{N_t, 1\}$ leads to smaller discounts and higher penalties, which is a prudent strategy for the insurer.

Also, the heterogeneity factor $\Theta$ could be replaced with a discrete approximation, like the $s$-convex minima obtained in DENUIT ET AL. (1998). All the integrals then reduce to sums of just a few terms. For instance, let us replace $\Theta$ with the 3-point random variable $\widetilde{\Theta}$ with support $\{0, t_-, t_+\}$ and corresponding probability masses $1 - q_- - q_+$, $q_-$ and $q_+$, given by

$$t_+ = \frac{\mu_1\mu_4 - \mu_2\mu_3 + \sqrt{(\mu_1\mu_4 - \mu_2\mu_3)^2 - 4(\mu_1\mu_3 - \mu_2^2)(\mu_2\mu_4 - \mu_3^2)}}{2(\mu_1\mu_3 - \mu_2^2)}$$

$$t_- = \frac{\mu_1\mu_4 - \mu_2\mu_3 - \sqrt{(\mu_1\mu_4 - \mu_2\mu_3)^2 - 4(\mu_1\mu_3 - \mu_2^2)(\mu_2\mu_4 - \mu_3^2)}}{2(\mu_1\mu_3 - \mu_2^2)}$$

$$q_+ = \frac{\mu_2 - t_-\mu_1}{t_+(t_+ - t_-)} \text{ and } q_- = \frac{\mu_2 - t_+\mu_1}{t_-(t_- - t_+)},$$

where $\mu_k = \mathbb{E}[\Theta^k]$, $k = 1, 2, 3, 4$. The random variable $\widetilde{\Theta}$ is the 5-convex minimum of DENUIT ET AL. (1999). We then get the results displayed in Table 4.1

| $t$ | $\delta$ | $\epsilon$ |
|---|---|---|
| 1 | 0.6161 | 0.1425 |
| 2 | 0.4649 | 0.0959 |
| 3 | 0.3776 | 0.0740 |
| 4 | 0.3200 | 0.0610 |
| 10 | 0.1729 | 0.0319 |
| 20 | 0.1006 | 0.0190 |
| 30 | 0.0715 | 0.0138 |

Table 4.1: Parameters $\delta_t$ and $\epsilon_t$ for different values of $t$ obtained with the discrete approximation $\widetilde{\Theta}$ to $\Theta$.

Compared with Table 2.1, we see that the differences in the $\epsilon_t$'s and $\delta_t$'s increase with $t$. The accuracy of the discrete approximation $\widetilde{\Theta}$ to $\Theta$ thus deteriorates with the length of the time horizon. For small $t$, replacing $\Theta$ with $\widetilde{\Theta}$ has almost no impact.

Averaging with respect to the age structure gives $\delta = 0.0872$ and $\epsilon = 0.0189$ (to be compared with 0.0710 and 0.0133). The discrepancies between the CRM coefficients obtained with $\Theta$ and with $\widetilde{\Theta}$ can be explained from the fact that the $\Psi_t(\epsilon, \delta)$ associated with large $t$'s play the prominent role in the determination of the unique $\epsilon - \delta$ pair.

The analysis conducted in this paper disregards a priori risk classification. An extension in the spirit of PITREBOIS ET AL. (2003) will be the topic of a future work. As pointed out in Remark 2.1, the possible correlation of $A$ with ratemaking variables and the dynamics of policyholders' characteristics require a careful examination.

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# Appendix: Main structure of the SAS/IML procedures

```
/* An analysis of the French BM System : computation of the CRM's at time t by solving
a system of 2 equations */

proc iml;

t = 3;

/* Parameters */
a = 0.8888;
lambda = 0.1474;

/*...*/

/* Function computing the sum of the squared differences between the left-hand side */
/* and the right-hand side of the two equations */
/* ------------------------------------------------------------ */
start quad_err(param2) global(param,lambda,t);
param = param2;
points = 0 100;
call quad(left1,”f_left1”,points);
/* f_left1 is a function computing the left-hand side of the first equation */
call quad(left2,”f_left2”,points);
/* f_left2 is a function computing the left-hand side of the second equation */
call quad(right1,”f_right1”,points);
/* f_right1 is a function computing the right-hand side of the first equation */
call quad(right2,”f_right2”,points);
/* f_right2 is a function computing the right-hand side of the second equation */
result = (right1-left1)**2+(right2-left2)**2;
return(result);
finish;
```

```
/* Main program : minimisation of the sum of the squared differences between the */
/* left-hand side and the right-hand side of the two equations */
/* ———————————————————————————— */
opt_max = 0 1;
start = 0.06 0.01;
call nlptr(rc,param2,”quad_err”,start) opt = opt_max;
delta = param2[1];
epsilon = param2[2];

quit;


/* An analysis of the French BM System : computation of the global CRM’s by solving a
system of 2 equations */

proc iml;

/* Parameters */
/* ———- */
a = 0.8888;
lambda = 0.1474;
w = 10 0 0 0 20 0 0 0 0 0 0 30 0 0 0 0 0 0 0 20 0 0 0 0 10 0 0 0 0 10;
w = w/100;

/* ... */

/* Function computing the sum of the squared differences between the left-hand side */
/* and the right-hand side of the two equations */
/* ———————————————————————————— */
start quad_err(param2) global(param,lambda,t,w);
param = param2;
points = 0 100;
l1=0;l2=0;r1=0;r2=0;
do t = 1 to ncol(w);
call quad(left1,”f_left1”,points);
/* f_left1 is a function computing the left-hand side of the first equation */
call quad(left2,”f_left2”,points);
/* f_left2 is a function computing the left-hand side of the second equation */
call quad(right1,”f_right1”,points);
/* f_right1 is a function computing the right-hand side of the first equation */
call quad(right2,”f_right2”,points);
/* f_right2 is a function computing the right-hand side of the second equation */
l1=l1+w[t]*t*left1;l2=l2+w[t]*t*left2;r1=r1+w[t]*t*right1;r2=r2+w[t]*t*right2;
end;
result = (r1-l1)**2+(r2-l2)**2;
```

```
return(result);
finish;

/* Main program : minimisation of the sum of the squared differences between the */
/* left-hand side and the right-hand side of the two equations */
/* ————————————————————————————————————————— */
opt_max = 0 1;
start = 0.07 0.013;
call nlptr(rc,param2,”quad_err”,start)opt = opt_max;
delta = param2[1];
epsilon = param2[2];

quit;
```