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ABSTRACT. Let $\{X_1, X_2, \dots\}$ be a sequence of independent and identically distributed positive random variables of Pareto-type and let $\{N(t);\, t \geq 0\}$ be a mixed Poisson process independent of the X_i's. For any fixed $t \geq 0$, define:

$$T_{N(t)} := \frac{X_1^2 + X_2^2 + \cdots + X_{N(t)}^2}{\left(X_1 + X_2 + \cdots + X_{N(t)}\right)^2}$$

if $N(t) \geq 1$ and $T_{N(t)} := 0$ otherwise. We determine the asymptotic behavior of any moment $\mathbb{E}\big[T_{N(t)}^k\big]$ as $t \to \infty$ with $k \in \mathbb{N}$. Our method relies on the theory of functions of regular variation and an integral representation of these moments.

1. INTRODUCTION

Let $\{X_1, X_2, \dots\}$ be a sequence of independent and identically distributed positive random variables with distribution function F and let $\{N(t);\, t \geq 0\}$ be a counting process independent of the X_i's. For any fixed $t \geq 0$, define:

$$T_{N(t)} := \frac{X_1^2 + X_2^2 + \cdots + X_{N(t)}^2}{\left(X_1 + X_2 + \cdots + X_{N(t)}\right)^2}$$

if $N(t) \geq 1$ and $T_{N(t)} := 0$ otherwise.

Denote by T_n the random variable $T_{N(t)}$ when the counting process $\{N(t);\, t \geq 0\}$ is non-random. An asymptotic analysis of T_n is provided by Albrecher et al. [1] and Albrecher and Teugels [2] under the condition that the distribution function F of X_1 is of *Pareto-type* with index $\alpha > 0$. To be more precise, Albrecher and Teugels [2] study the asymptotic behavior of arbitrary moments of T_n as $n \to \infty$, generalizing earlier results pertaining to $\mathbb{E}[T_n]$ by Fuchs et al. [6], and Albrecher et al. [1] derive limit distributions for the properly normalized quantity T_n as $n \to \infty$.

This paper focuses on *moment convergence*. We establish the asymptotic behavior of any moment of order $k \in \mathbb{N}$ of the ratio $T_{N(t)}$ as $t \to \infty$ under the conditions that the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$ and that the counting process $\{N(t);\, t \geq 0\}$ is *mixed Poisson*. The appropriate definitions are recalled in Section 2 along with some properties that will prove to be useful later on. The results of the paper rely on the theory of *functions of regular variation* (e.g., Bingham et al. [4]) and an integral representation of $\mathbb{E}\big[T_{N(t)}^k\big]$ in terms of the probability generating function of $N(t)$ and the *Laplace transform* of X_1, following in that the basis for the analysis in Albrecher and Teugels [2].

Let μ_β denote the moment of order $\beta > 0$ of X_1, i.e.:

$$\mu_\beta := \mathbb{E}\big[X_1^\beta\big] = \beta \int_0^\infty x^{\beta-1}\,(1 - F(x))\,dx \leq \infty.$$

As pointed out by Albrecher and Teugels [2], both the numerator and the denominator defining $T_{N(t)}$ exhibit an erratic behavior if $\mu_1 = \infty$, whereas this is the case only for the numerator if $\mu_1 < \infty$ and $\mu_2 = \infty$. When X_1 (or equivalently F) is of Pareto-type with index $\alpha > 0$, it turns out that μ_β is finite if $\beta < \alpha$ but infinite whenever $\beta > \alpha$. In particular, $\mu_1 < \infty$ if $\alpha > 1$ while $\mu_2 < \infty$ as soon as $\alpha > 2$. Since the asymptotic behavior of $T_{N(t)}$ as $t \to \infty$ is influenced by the (non)finiteness of μ_1 and/or μ_2, different kinds of results will then show up according to the range of α. This is expressed in our main results in

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Section 3. Some concluding remarks are given in Section 4. In particular, we point out the connection between $T_{N(t)}$ and the *sample coefficient of variation* of a random sample $X_1, \dots, X_{N(t)}$ from a positive random variable X of random size $N(t)$ from a nonnegative integer-valued distribution.

2. Preliminaries

Though standard notations, we mention that $\xrightarrow{a.s.}$ stands for almost sure convergence. For two measurable functions f and g, we write $f(x) = o(g(x))$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 0$ and $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. Also, $\Gamma(\cdot)$ denotes the gamma function and $B(\cdot,\cdot)$ denotes the beta function. Finally, the set of nonnegative integers is denoted by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Recall that a counting process $\{N(t);\, t \geq 0\}$ is said to be a mixed Poisson process if there exists a positive random variable Λ such that $\{N(t);\, t \geq 0\} = \{\tilde{N}(\Lambda t);\, t \geq 0\}$, where $\{\tilde{N}(t);\, t \geq 0\}$ is a homogeneous Poisson process with intensity 1 independent of Λ. For each fixed $t \geq 0$, the random variable $N(t)$ has a mixed Poisson distribution given by:

$$p_n(t) := \mathbb{P}[N(t) = n] = \mathbb{E}\left[\frac{(\Lambda t)^n}{n!}\, e^{-\Lambda t}\right] = \int_0^\infty \frac{(\lambda t)^n}{n!}\, e^{-\lambda t}\, dH(\lambda), \quad n \in \mathbb{N}_0$$

where H denotes the distribution function of the mixing random variable Λ and is called the mixing distribution. The role of the mixing random variable Λ can be highlighted by observing that:

$$\frac{N(t)}{t} \xrightarrow{a.s.} \Lambda \quad \text{as} \quad t \to \infty.$$

When the mixing distribution H is degenerate at a single point $\lambda \in (0, \infty)$, we retrieve the homogeneous Poisson process with intensity λ. The latter plays a crucial role in practical applications. In particular, it is the most popular among all claim number processes in the actuarial literature. The mixed Poisson process, introduced to actuaries by Dubourdieu [5], has always been very popular among (re)insurance modelers. It has found many applications in (re)insurance mathematics because of its flexibility, its success in actuarial data fitting and its property of being more dispersed than the Poisson process. For a general overview on mixed Poisson processes, we refer to the monograph by Grandell [7].

For a fixed $t \geq 0$, the probability generating function of $N(t)$ is denoted by Q_t and satisfies:

$$Q_t(z) := \mathbb{E}\big[z^{N(t)}\big] = \sum_{n=0}^{\infty} p_n(t)\, z^n = \mathbb{E}\big[e^{-t(1-z)\Lambda}\big], \quad |z| \leq 1.$$

For every $r \in \mathbb{N}_0$, the rth derivative of $Q_t(z)$ with respect to z is denoted by $Q_t^{(r)}(z)$ and defined for $|z| < 1$. It can be expressed in terms of expectations as:

$$Q_t^{(r)}(z) = r!\, \mathbb{E}\left[\binom{N(t)}{r} z^{N(t)-r}\right] = t^r\, \mathbb{E}\big[e^{-t(1-z)\Lambda} \Lambda^r\big].$$

We define the auxiliary quantities $q_r(w) := \mathbb{E}[e^{-w\Lambda}\Lambda^r]$ for all $w \geq 0$ and $r \in \mathbb{N}_0$. Note that $q_r(0) = \mathbb{E}[\Lambda^r] \leq \infty$. For all $w \in [0, t]$ and $r \in \mathbb{N}_0$, the following identity then holds:

$$\frac{1}{t^r}\, Q_t^{(r)}(1 - w/t) = q_r(w)$$

where the right-hand side does no longer depend upon t.

Before giving an easy but useful result on the moment condition for the mixing random variable Λ, notice that for any $\beta > 0$ and $r \in \mathbb{N}_0$:

$$\int_0^\infty w^{\beta-1}\, q_r(w)\, dw = \Gamma(\beta)\, \mathbb{E}\big[\Lambda^{r-\beta}\big]. \tag{1}$$

Lemma 1. *Let Λ be a positive random variable with distribution function H. Then for all $0 < r \leq s$, $\mathbb{E}[\Lambda^s] < \infty \implies \mathbb{E}[\Lambda^r] < \infty$ and $\mathbb{E}[\Lambda^{-s}] < \infty \implies \mathbb{E}[\Lambda^{-r}] < \infty$.*

Proof. Let $0 < r \leq s$ be fixed. Assume that $\mathbb{E}[\Lambda^s] < \infty$. Then:

$$\mathbb{E}[\Lambda^r] = \int_0^1 \lambda^r\, dH(\lambda) + \int_1^\infty \lambda^r\, dH(\lambda) \leq \mathbb{P}[\Lambda \leq 1] + \mathbb{E}[\Lambda^s] < \infty.$$

Now, assume that $\mathbb{E}[\Lambda^{-s}] < \infty$. Then:

$$\mathbb{E}[\Lambda^{-r}] = \int_0^1 \lambda^{-r}\, dH(\lambda) + \int_1^\infty \lambda^{-r}\, dH(\lambda) \leq \mathbb{E}[\Lambda^{-s}] + \mathbb{P}[\Lambda > 1] < \infty$$

and the claim is proved. □

Let us turn to the distribution function F of X_1. Our asymptotic results are derived under the condition that F is of Pareto-type with index $\alpha > 0$. This means that $1 - F$ is regularly varying at ∞ with index $-\alpha < 0$, i.e.:

$$1 - F(x) \sim x^{-\alpha}\ell(x) \quad \text{as} \quad x \to \infty \tag{2}$$

where the function ℓ is slowly varying at ∞.

Recall that a measurable function $f\colon (0,\infty) \to (0,\infty)$ is regularly varying at ∞ with index $\gamma \in \mathbb{R}$ (written $f \in \mathrm{RV}_\gamma^\infty$) if for all $x > 0$, $\lim_{t\to\infty} f(tx)/f(t) = x^\gamma$. When $\gamma = 0$, f is said to be slowly varying at ∞. Similarly, a measurable function $g\colon (0,\infty) \to (0,\infty)$ is regularly varying at 0 with index $\gamma \in \mathbb{R}$ (written $g \in \mathrm{RV}_\gamma^0$) if for all $x > 0$, $\lim_{s\downarrow 0} g(sx)/g(s) = x^\gamma$. When $\gamma = 0$, g is said to be slowly varying at 0. We refer to Bingham et al. [4] for a textbook treatment on the theory of functions of regular variation.

It is well-known that the tail condition (2) appears in extreme value theory as the essential condition in the Fréchet-Pareto domain of attraction problem. For a recent treatment, see Beirlant et al. [3]. When $\alpha \in (0,2)$, the condition is necessary and sufficient for F to belong to the additive domain of attraction of a non-normal stable law with exponent α (e.g., Theorem 8.3.1 of Bingham et al. [4]).

The common Laplace transform of the X_i's is defined and denoted by:

$$\varphi(s) := \mathbb{E}\big[e^{-sX_1}\big] = \int_0^\infty e^{-sx}\, dF(x), \quad s \geq 0.$$

For every $n \in \mathbb{N}$, we denote by $\varphi^{(n)}(s)$ the nth derivative of $\varphi(s)$ with respect to s. By Lemma 3.1 of Albrecher and Teugels [2] and Bingham-Doney's lemma (e.g., Theorem 8.1.6 of Bingham et al. [4]), the asymptotic behavior of $\varphi^{(n)}$ at the origin when F satisfies (2) is the following.

Lemma 2. *Assume that the distribution function F of X_1 satisfies $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\alpha > 0$ and $\ell \in \mathrm{RV}_0^\infty$. Then for any $n \in \mathbb{N}$, we have as $s \downarrow 0$:*

$$(-1)^n\, \varphi^{(n)}(s) \sim \begin{cases} \alpha\,\Gamma(n-\alpha)\, s^{\alpha-n}\, \ell(1/s) & \text{if } n > \alpha \\ \alpha\, \tilde{\ell}(1/s) & \text{if } n = \alpha \text{ and } \mu_n = \infty \\ \mu_n & \text{if } n < \alpha \text{ or if } n = \alpha \text{ and } \mu_n < \infty \end{cases}$$

where $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, du \in \mathrm{RV}_0^\infty$.

Now, we give our results.

3. Results

We start by deriving an integral representation for the kth moment of $T_{N(t)}$ for fixed $t \geq 0$ and $k \in \mathbb{N}$. Note that we do not make any specific assumption on the distribution function F of X_1 and on the counting process $\{N(t);\, t \geq 0\}$.

Lemma 3. *Let $t \geq 0$ and $k \in \mathbb{N}$ be fixed. The kth moment of $T_{N(t)}$ is then given by:*

$$\mathbb{E}\big[T_{N(t)}^k\big] = \sum_{r=1}^k \sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r \\ k_1+\cdots+k_r=k}} \frac{k!}{\prod_{i=1}^r k_i!}\, \frac{B_t(k_1,\dots,k_r)}{(2k-1)!\, r!} \tag{3}$$

with:

$$B_t(k_1,\dots,k_r) := \int_0^\infty s^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)}(s)\, Q_t^{(r)}(\varphi(s))\, ds. \tag{4}$$

Proof. Let $t \geq 0$ and $k \in \mathbb{N}$ be fixed. For each $n \in \mathbb{N}$, we interpret $\binom{n}{r} := 0$ whenever $r \geq n+1$. Using Lemma 2.1 of Albrecher and Teugels [2], we easily derive:

$$\mathbb{E}\big[T_{N(t)}^k\big] = \sum_{n=0}^\infty p_n(t)\, \mathbb{E}\big[T_{N(t)}^k \big| N(t) = n\big] = \sum_{n=1}^\infty p_n(t)\, \mathbb{E}\big[T_n^k\big]$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} p_n(t) \sum_{r=1}^{k} \sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r \\ k_1+\cdots+k_r=k}} \frac{k!}{\prod_{i=1}^r k_i!} \frac{\binom{n}{r}}{(2k-1)!} \int_0^\infty s^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)}(s)\, \varphi^{n-r}(s)\, ds \\
&= \sum_{r=1}^{k} \sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r \\ k_1+\cdots+k_r=k}} \frac{k!}{\prod_{i=1}^r k_i!} \frac{1}{(2k-1)!} \int_0^\infty s^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)}(s) \sum_{n=r}^{\infty} \binom{n}{r} p_n(t)\, \varphi^{n-r}(s)\, ds \\
&= \sum_{r=1}^{k} \sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r \\ k_1+\cdots+k_r=k}} \frac{k!}{\prod_{i=1}^r k_i!} \frac{1}{(2k-1)!\, r!} \int_0^\infty s^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)}(s)\, Q_t^{(r)}(\varphi(s))\, ds
\end{aligned}
$$

and the proof is finished. □

We henceforth assume that $\{N(t);\, t \ge 0\}$ is a mixed Poisson process with mixing random variable Λ. We rewrite $B_t(k_1,\dots,k_r)$ in a more convenient form which will prove to be very useful. Defining $\psi(s) := \varphi^{-1}(1-s)$ for $s \in [0,1)$ and substituting $s = \psi(w/t)$ in (4) leads for every $t > 0$ to:

$$
\begin{aligned}
&B_t(k_1,\dots,k_r) \\
&= \int_0^\infty s^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)}(s)\, Q_t^{(r)}(\varphi(s))\, ds \\
&= \int_0^t \psi^{2k-1}(w/t) \prod_{i=1}^r \varphi^{(2k_i)}(\psi(w/t))\, Q_t^{(r)}(1-w/t)\, d_w\psi(w/t) \\
&= -t^{r-1} \frac{\psi^{2k-1}(1/t)}{\varphi^{(1)}(\psi(1/t))} \prod_{i=1}^r \varphi^{(2k_i)}(\psi(1/t)) \int_0^t \left(\frac{\psi(w/t)}{\psi(1/t)}\right)^{2k-1} \prod_{i=1}^r \frac{\varphi^{(2k_i)}(\psi(w/t))}{\varphi^{(2k_i)}(\psi(1/t))} \frac{\varphi^{(1)}(\psi(1/t))}{\varphi^{(1)}(\psi(w/t))} \frac{Q_t^{(r)}(1-w/t)}{t^r}\, dw \\
&= -t^{r-1} \frac{\psi^{2k-1}(1/t)}{\varphi^{(1)}(\psi(1/t))} \prod_{i=1}^r \varphi^{(2k_i)}(\psi(1/t)) \int_0^\infty \left(\frac{\psi(w/t)}{\psi(1/t)}\right)^{2k-1} \prod_{i=1}^r \frac{\varphi^{(2k_i)}(\psi(w/t))}{\varphi^{(2k_i)}(\psi(1/t))} \frac{\varphi^{(1)}(\psi(1/t))}{\varphi^{(1)}(\psi(w/t))}\, q_r(w)\, \mathbb{1}_{(0,t)}(w)\, dw \\
&= f_t(k_1,\dots,k_r) \int_0^\infty g_t(w;k_1,\dots,k_r)\, dw
\end{aligned}
$$

with:

$$
f_t(k_1,\dots,k_r) := -t^{r-1} \frac{\psi^{2k-1}(1/t)}{\varphi^{(1)}(\psi(1/t))} \prod_{i=1}^r \varphi^{(2k_i)}(\psi(1/t))
$$

and:

$$
g_t(w;k_1,\dots,k_r) := \left(\frac{\psi(w/t)}{\psi(1/t)}\right)^{2k-1} \prod_{i=1}^r \frac{\varphi^{(2k_i)}(\psi(w/t))}{\varphi^{(2k_i)}(\psi(1/t))} \frac{\varphi^{(1)}(\psi(1/t))}{\varphi^{(1)}(\psi(w/t))}\, q_r(w)\, \mathbb{1}_{(0,t)}(w), \quad w \ge 0.
$$

From now on, we assume that X_1 is of Pareto-type with index $\alpha > 0$, i.e. that F satisfies (2) for some $\ell \in \mathrm{RV}_0^\infty$. Here is the first of our main results pertaining to moment convergence for $T_{N(t)}$. It concerns the case $\alpha \in (0,1)$.

Theorem 1. *Assume that X_1 is of Pareto-type with index $\alpha \in (0,1)$ and that $\{N(t);\, t\ge 0\}$ is a mixed Poisson process with mixing random variable Λ. If $\mathbb{E}[\Lambda^\epsilon] < \infty$ and $\mathbb{E}[\Lambda^{-\epsilon}] < \infty$ for some $\epsilon > 0$, then for any fixed $k \in \mathbb{N}$:*

$$
\lim_{t\to\infty} \mathbb{E}\big[T_{N(t)}^k\big] = \frac{k!}{(2k-1)!} \sum_{r=1}^k \frac{\alpha^{r-1}}{r\,\Gamma^r(1-\alpha)}\, G(r,k)
$$

where $G(r,k)$ is the coefficient of x^k in the polynomial $\left(\sum_{i=1}^{k-r+1} \frac{\Gamma(2i-\alpha)}{i!}\, x^i\right)^r$.

Proof. Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$ be fixed. Since $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0^\infty$, it follows from Corollary 8.1.7 of Bingham et al. [4] that $1 - \varphi(s) \sim \Gamma(1-\alpha)\, s^\alpha \ell(1/s)$ as $s \downarrow 0$. Hence, we easily deduce $s = 1 - \varphi(\psi(s)) \sim \Gamma(1-\alpha)\, \psi^\alpha(s)\, \ell(1/\psi(s))$ as $s \downarrow 0$ which leads to:

$$
\lim_{s\downarrow 0} s^{-1}\, \psi^\alpha(s)\, \ell(1/\psi(s)) = \frac{1}{\Gamma(1-\alpha)}. \tag{5}
$$

First, we determine the asymptotic behavior of $B_t(k_1,\dots,k_r)$ as $t \to \infty$ for any fixed integer $1 \le r \le k$. Relation (5) learns us that $\psi \in \mathrm{RV}_{1/\alpha}^0$, leading to $\lim_{s\downarrow 0} \psi(s) = 0$. For $i = 1,\dots,r$, we then have $\varphi^{(2k_i)} \circ \psi \in \mathrm{RV}_{(\alpha-2k_i)/\alpha}^0$ since $\varphi^{(2k_i)} \in \mathrm{RV}_{\alpha-2k_i}^0$ by Lemma 2. Moreover, $-\varphi^{(1)} \circ \psi \in \mathrm{RV}_{(\alpha-1)/\alpha}^0$ since

$-\varphi^{(1)} \in \mathrm{RV}^0_{\alpha-1}$ by Lemma 2. Set $\delta_r := \frac{\zeta}{2k+r}$ with $\zeta = \epsilon$ if $\epsilon \in (0,1)$ or $\zeta \in (0,1)$ otherwise. Thus, for this chosen $\delta_r > 0$, there exists $C_r = C_r(\delta_r) > 1$ such that for $t > 0$ and $w > 0$:

$$\begin{aligned} g_t(w;k_1,\ldots,k_r) &\leq C_r\, w^{\frac{2k-1}{\alpha}} \big(\max\{w^{\delta_r}, w^{-\delta_r}\}\big)^{2k-1} w^{\frac{r\alpha-2k}{\alpha}} \big(\max\{w^{\delta_r}, w^{-\delta_r}\}\big)^{r+1} w^{\frac{1-\alpha}{\alpha}} q_r(w) \\ &= C_r\, w^{r-1} \max\{w^{\zeta}, w^{-\zeta}\}\, q_r(w) =: h(w) \end{aligned}$$

by virtue of Potter's theorem (e.g., Theorem 1.5.6 of Bingham et al. [4]). Now, $\int_0^\infty h(w)\,dw < \infty$ if and only if $\int_0^1 w^{r-1-\zeta}\, q_r(w)\,dw < \infty$ and $\int_1^\infty w^{r-1+\zeta}\, q_r(w)\,dw < \infty$. Since $\zeta \in (0,1)$ and $\zeta \leq \epsilon$, we use (1) together with Lemma 1 to get:

$$\int_0^1 w^{r-1-\zeta}\, q_r(w)\,dw \leq \int_0^\infty w^{r-1-\zeta}\, q_r(w)\,dw = \Gamma(r-\zeta)\,\mathbb{E}[\Lambda^{\zeta}] < \infty$$

and:

$$\int_1^\infty w^{r-1+\zeta}\, q_r(w)\,dw \leq \int_0^\infty w^{r-1+\zeta}\, q_r(w)\,dw = \Gamma(r+\zeta)\,\mathbb{E}[\Lambda^{-\zeta}] < \infty.$$

Hence, the function h is integrable. Finally, $\lim_{t\to\infty} g_t(w;k_1,\ldots,k_r) = w^{r-1}q_r(w)$ for every $w > 0$. Thus, applying Lebesgue's theorem on dominated convergence and using (1), we deduce:

$$\lim_{t\to\infty} \int_0^\infty g_t(w;k_1,\ldots,k_r)\,dw = \int_0^\infty w^{r-1}\, q_r(w)\,dw = (r-1)!.$$

Using Lemma 2 leads as $t \to \infty$ to:

$$\begin{aligned} f_t(k_1,\ldots,k_r) &\sim \frac{-\alpha^r \prod_{i=1}^r \Gamma(2k_i-\alpha)\, t^{r-1}\, \psi^{r\alpha-1}(1/t)\, \ell^r\big(1/\psi\big(\frac{1}{t}\big)\big)}{-\alpha\,\Gamma(1-\alpha)\, \psi^{\alpha-1}(1/t)\, \ell\big(1/\psi\big(\frac{1}{t}\big)\big)} \\ &= \alpha^{r-1} \frac{\prod_{i=1}^r \Gamma(2k_i-\alpha)}{\Gamma(1-\alpha)}\, t^{r-1}\, \psi^{\alpha(r-1)}(1/t)\, \ell^{r-1}(1/\psi(1/t)) \end{aligned}$$

so that we get by using the relation (5):

$$\lim_{t\to\infty} f_t(k_1,\ldots,k_r) = \frac{\alpha^{r-1}\prod_{i=1}^r \Gamma(2k_i-\alpha)}{\Gamma^r(1-\alpha)}.$$

Therefore, we obtain:

$$\lim_{t\to\infty} B_t(k_1,\ldots,k_r) = \frac{(r-1)!\,\alpha^{r-1}\prod_{i=1}^r \Gamma(2k_i-\alpha)}{\Gamma^r(1-\alpha)}.$$

Summing over all $r = 1,\ldots,k$ in (3), we finally arrive at:

$$\lim_{t\to\infty} \mathbb{E}\big[T^k_{N(t)}\big] = \frac{k!}{(2k-1)!} \sum_{r=1}^k \frac{\alpha^{r-1}}{r\,\Gamma^r(1-\alpha)} \sum_{\substack{(k_1,\ldots,k_r)\in\mathbb{N}^r \\ k_1+\cdots+k_r=k}} \prod_{i=1}^r \frac{\Gamma(2k_i-\alpha)}{k_i!}.$$

The theorem is proved since Albrecher and Teugels [2] have observed that:

$$G(r,k) := \sum_{\substack{(k_1,\ldots,k_r)\in\mathbb{N}^r \\ k_1+\cdots+k_r=k}} \prod_{i=1}^r \frac{\Gamma(2k_i-\alpha)}{k_i!}$$

can be read off as the coefficient of x^k in the r-fold product $\left(\sum_{i=1}^{k-r+1} \frac{\Gamma(2i-\alpha)}{i!}\, x^i\right)^r$. □

Our next result deals with the case $\alpha = 1$ if $\mu_1 = \infty$.

Theorem 2. *Assume that X_1 is of Pareto-type with index $\alpha = 1$ and $\mu_1 = \infty$. Assume that $\{N(t);\, t \geq 0\}$ is a mixed Poisson process with mixing random variable Λ. If $\mathbb{E}[\Lambda^{\epsilon}] < \infty$ and $\mathbb{E}[\Lambda^{-\epsilon}] < \infty$ for some $\epsilon > 0$, then for any fixed $k \in \mathbb{N}$:*

$$\mathbb{E}\big[T^k_{N(t)}\big] \sim \frac{1}{2k-1} \frac{\ell(a_t)}{\tilde{\ell}(a_t)} \quad as \quad t \to \infty$$

where $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,du \in \mathrm{RV}^\infty_0$ and $(a_t)_{t>0}$ is a sequence defined by $\lim_{t\to\infty} t\, a_t^{-1}\, \tilde{\ell}(a_t) = 1$.

Proof. Let $k \in \mathbb{N}$ be fixed. Since $1 - F(x) \sim x^{-1}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0^\infty$ such that $\mu_1 = \infty$, it follows from Lemma 2 that $\varphi^{(1)}(s) \sim -\tilde{\ell}(1/s)$ as $s \downarrow 0$ and then that $1 - \varphi(s) \sim s\,\tilde{\ell}(1/s)$ as $s \downarrow 0$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,du \in \mathrm{RV}_0^\infty$. Since $s = 1 - \varphi(\psi(s)) \sim \psi(s)\,\tilde{\ell}(1/\psi(s))$ as $s \downarrow 0$, we obtain:

$$\lim_{s\downarrow 0} s^{-1}\,\psi(s)\,\tilde{\ell}(1/\psi(s)) = 1. \tag{6}$$

First, we determine the asymptotic behavior of $B_t(k_1, \dots, k_r)$ as $t \to \infty$ for any fixed integer $1 \le r \le k$. Relation (6) learns us that $\psi \in \mathrm{RV}_1^0$, leading to $\lim_{s\downarrow 0}\psi(s) = 0$. For $i = 1, \dots, r$, we then have $\varphi^{(2k_i)} \circ \psi \in \mathrm{RV}_{1-2k_i}^0$ since $\varphi^{(2k_i)} \in \mathrm{RV}_{1-2k_i}^0$ by Lemma 2. Moreover, $-\varphi^{(1)} \circ \psi \in \mathrm{RV}_0^0$ since $-\varphi^{(1)} \in \mathrm{RV}_0^0$. Set $\delta_r := \frac{\zeta}{2k+r}$ with $\zeta = \epsilon$ if $\epsilon \in (0,1)$ or $\zeta \in (0,1)$ otherwise. Thus, for this chosen $\delta_r > 0$, there exists $C_r = C_r(\delta_r) > 1$ such that for $t > 0$ and $w > 0$:

$$\begin{aligned} g_t(w; k_1, \dots, k_r) &\le C_r\, w^{2k-1}\big(\max\{w^{\delta_r}, w^{-\delta_r}\}\big)^{2k-1}\, w^{r-2k}\big(\max\{w^{\delta_r}, w^{-\delta_r}\}\big)^{r+1}\, q_r(w) \\ &= C_r\, w^{r-1}\max\{w^{\zeta}, w^{-\zeta}\}\, q_r(w) =: h(w) \end{aligned}$$

by virtue of Potter's theorem. Now, $\int_0^\infty h(w)\,dw < \infty$ if and only if $\int_0^1 w^{r-1-\zeta}\,q_r(w)\,dw < \infty$ and $\int_1^\infty w^{r-1+\zeta}\,q_r(w)\,dw < \infty$. Since $\zeta \in (0,1)$ and $\zeta \le \epsilon$, we use (1) together with Lemma 1 to get:

$$\int_0^1 w^{r-1-\zeta}\,q_r(w)\,dw \le \int_0^\infty w^{r-1-\zeta}\,q_r(w)\,dw = \Gamma(r-\zeta)\,\mathbb{E}[\Lambda^{\zeta}] < \infty$$

and:

$$\int_1^\infty w^{r-1+\zeta}\,q_r(w)\,dw \le \int_0^\infty w^{r-1+\zeta}\,q_r(w)\,dw = \Gamma(r+\zeta)\,\mathbb{E}[\Lambda^{-\zeta}] < \infty.$$

Hence, the function h is integrable. Finally, $\lim_{t\to\infty} g_t(w; k_1, \dots, k_r) = w^{r-1}q_r(w)$. Thus, applying Lebesgue's theorem on dominated convergence and using (1), we deduce:

$$\lim_{t\to\infty}\int_0^\infty g_t(w; k_1, \dots, k_r)\,dw = \int_0^\infty w^{r-1}\,q_r(w)\,dw = (r-1)!.$$

Using Lemma 2, relation (6) and $\varphi^{(1)}(\psi(s)) \sim -\tilde{\ell}(1/\psi(s))$ as $s \downarrow 0$, we get as $t \to \infty$:

$$f_t(k_1, \dots, k_r) \sim \prod_{i=1}^r \Gamma(2k_i - 1)\, t^{r-1}\,\psi^{r-1}(1/t)\frac{\ell^r\big(1/\psi\big(\frac{1}{t}\big)\big)}{\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big)} \sim \prod_{i=1}^r \Gamma(2k_i - 1)\left(\frac{\ell\big(1/\psi\big(\frac{1}{t}\big)\big)}{\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big)}\right)^r.$$

Therefore, we obtain:

$$B_t(k_1, \dots, k_r) \sim (r-1)!\prod_{i=1}^r \Gamma(2k_i - 1)\left(\frac{\ell\big(1/\psi\big(\frac{1}{t}\big)\big)}{\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big)}\right)^r \quad \text{as} \quad t \to \infty.$$

Since $\lim_{t\to\infty}\frac{\ell(1/\psi(\frac{1}{t}))}{\tilde{\ell}(1/\psi(\frac{1}{t}))} = 0$ (e.g., Proposition 1.5.9a of Bingham et al. [4]), only the summand with $r = 1$ contributes to the dominating asymptotic term of (3). Hence, we get:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{1}{2k-1}\frac{\ell\big(1/\psi\big(\frac{1}{t}\big)\big)}{\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big)} \quad \text{as} \quad t \to \infty.$$

One easily notes that the limit relation (6) is equivalent to $\lim_{t\to\infty} t\,\psi(1/t)\,\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big) = 1$. Defining a sequence $(a_t)_{t>0}$ by $\lim_{t\to\infty} t\,a_t^{-1}\tilde{\ell}(a_t) = 1$ implies that $\lim_{t\to\infty} a_t\,\psi(1/t) = 1$. By virtue of the uniform convergence theorem for slowly varying functions (e.g., Theorem 1.2.1 of Bingham et al. [4]), we thus get $\ell\big(1/\psi\big(\frac{1}{t}\big)\big) \sim \ell(a_t)$ and $\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big) \sim \tilde{\ell}(a_t)$ as $t \to \infty$. Consequently, we finally arrive at:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{1}{2k-1}\frac{\ell(a_t)}{\tilde{\ell}(a_t)} \quad \text{as} \quad t \to \infty$$

and the proof is finished. □

In the following result, the case $\alpha \in (1,2)$ (including $\alpha = 1$ if $\mu_1 < \infty$) is treated.

Theorem 3. *Assume that X_1 is of Pareto-type with index $\alpha \in (1,2)$ (including $\alpha = 1$ if $\mu_1 < \infty$) and that $\{N(t);\, t \geq 0\}$ is a mixed Poisson process with mixing random variable Λ. For any fixed $k \in \mathbb{N}$, if $\mathbb{E}[\Lambda^{\epsilon}] < \infty$ and $\mathbb{E}[\Lambda^{-(k(\alpha-1)+\epsilon)}] < \infty$ for some $\epsilon > 0$, then:*

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{\alpha}{\mu_1^{\alpha}}\, B(2k-\alpha,\alpha)\, \mathbb{E}[\Lambda^{-(\alpha-1)}]\, t^{-(\alpha-1)}\, \ell(t) \quad \text{as} \quad t \to \infty.$$

Proof. Let $k \in \mathbb{N}$ be fixed. Let $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0^{\infty}$ and $\alpha \in (1,2)$ or $\alpha = 1$ if $\mu_1 < \infty$.

First, we determine the asymptotic behavior of $B_t(k_1,\dots,k_r)$ as $t \to \infty$ for any fixed integer $1 \leq r \leq k$. Since $\mu_1 < \infty$, it follows that $\varphi^{(1)}(0) = -\mu_1$ and $1-\varphi(s) \sim \mu_1 s$ as $s \downarrow 0$. Hence, $s = 1-\varphi(\psi(s)) \sim \mu_1\psi(s)$ as $s \downarrow 0$ and we deduce that $\psi(s) \underset{s\downarrow 0}{\sim} s/\mu_1 \in \mathrm{RV}_1^0$, leading to $\lim_{s\downarrow 0}\psi(s) = 0$. For $i = 1,\dots,r$, we then have $\varphi^{(2k_i)} \circ \psi \in \mathrm{RV}_{\alpha-2k_i}^0$ since $\varphi^{(2k_i)} \in \mathrm{RV}_{\alpha-2k_i}^0$ by Lemma 2. Moreover, $-\varphi^{(1)} \circ \psi \in \mathrm{RV}_0^0$ since $-\varphi^{(1)}(s) \underset{s\downarrow 0}{\sim} \mu_1 \in \mathrm{RV}_0^0$. Set $\delta_r := \frac{\zeta}{2k+r}$ with $\zeta = \epsilon$ if $\epsilon \in (0,1)$ or $\zeta \in (0,1)$ otherwise. Thus, for this chosen $\delta_r > 0$, there exists $C_r = C_r(\delta_r) > 1$ such that for $t > 0$ and $w > 0$:

$$\begin{aligned} g_t(w;k_1,\dots,k_r) &\leq C_r\, w^{2k-1}\big(\max\{w^{\delta_r}, w^{-\delta_r}\}\big)^{2k-1}\, w^{r\alpha-2k}\big(\max\{w^{\delta_r}, w^{-\delta_r}\}\big)^{r+1}\, q_r(w) \\ &= C_r\, w^{r\alpha-1}\max\{w^{\zeta}, w^{-\zeta}\}\, q_r(w) =: h(w) \end{aligned}$$

by virtue of Potter's theorem. Now, $\int_0^{\infty} h(w)\,dw < \infty$ if and only if $\int_0^1 w^{r\alpha-1-\zeta}\, q_r(w)\,dw < \infty$ and $\int_1^{\infty} w^{r\alpha-1+\zeta}\, q_r(w)\,dw < \infty$. Since $\zeta \in (0,1)$ and $\zeta \leq \epsilon$, we use (1) together with Lemma 1 to get:

$$\int_0^1 w^{r\alpha-1-\zeta}\, q_r(w)\,dw \leq \int_0^1 w^{r-1-\zeta}\, q_r(w)\,dw \leq \int_0^{\infty} w^{r-1-\zeta}\, q_r(w)\,dw = \Gamma(r-\zeta)\,\mathbb{E}[\Lambda^{\zeta}] < \infty$$

and since $-(k(\alpha-1)+\epsilon) \leq -(r(\alpha-1)+\zeta) < 0$:

$$\int_1^{\infty} w^{r\alpha-1+\zeta}\, q_r(w)\,dw \leq \int_0^{\infty} w^{r\alpha-1+\zeta}\, q_r(w)\,dw = \Gamma(r\alpha+\zeta)\,\mathbb{E}[\Lambda^{-(r(\alpha-1)+\zeta)}] < \infty.$$

Hence, the function h is integrable. Finally, $\lim_{t\to\infty} g_t(w;k_1,\dots,k_r) = w^{r\alpha-1}q_r(w)$ for every $w > 0$. Thus, applying Lebesgue's theorem on dominated convergence and using (1), we deduce:

$$\lim_{t\to\infty}\int_0^{\infty} g_t(w;k_1,\dots,k_r)\,dw = \int_0^{\infty} w^{r\alpha-1}\, q_r(w)\,dw = \Gamma(r\alpha)\,\mathbb{E}[\Lambda^{-r(\alpha-1)}].$$

Since $\ell(1/s) \in \mathrm{RV}_0^0$, the uniform convergence theorem for slowly varying functions states that $\ell(x/s) \sim \ell(1/s)$ as $s \downarrow 0$ uniformly on each compact x-set in $(0,\infty)$. Since $\lim_{s\downarrow 0} s/\psi(s) = \mu_1 \in (0,\infty)$, we then get $\ell(1/\psi(s)) = \ell\Big(\frac{s}{\psi(s)}\frac{1}{s}\Big) \sim \ell(1/s)$ as $s \downarrow 0$. This together with Lemma 2 and $\lim_{s\downarrow 0}\varphi^{(1)}(\psi(s)) = -\mu_1$ yields as $t \to \infty$:

$$f_t(k_1,\dots,k_r) \sim \frac{\alpha^r}{\mu_1}\prod_{i=1}^{r}\Gamma(2k_i-\alpha)\, t^{r-1}\, \psi^{r\alpha-1}(1/t)\, \ell^r(1/\psi(1/t)) \sim \frac{\alpha^r}{\mu_1^{r\alpha}}\prod_{i=1}^{r}\Gamma(2k_i-\alpha)\, t^{-r(\alpha-1)}\, \ell^r(t).$$

Therefore, we obtain:

$$B_t(k_1,\dots,k_r) \sim \left(\frac{\alpha}{\mu_1^{\alpha}}\right)^r \Gamma(r\alpha)\prod_{i=1}^{r}\Gamma(2k_i-\alpha)\,\mathbb{E}[\Lambda^{-r(\alpha-1)}]\,\Big(t^{-(\alpha-1)}\,\ell(t)\Big)^r \quad \text{as} \quad t \to \infty.$$

Note that when $\alpha = 1$, we have $\ell(x) = o(1)$ as $x \to \infty$ because $\mu_1 < \infty$. Since $\lim_{t\to\infty} t^{-(\alpha-1)}\ell(t) = 0$, the first-order asymptotic behavior of (3) is solely determined by the term with $r = 1$. Hence, we obtain:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{\alpha}{\mu_1^{\alpha}}\,\frac{\Gamma(2k-\alpha)\Gamma(\alpha)}{(2k-1)!}\,\mathbb{E}[\Lambda^{-(\alpha-1)}]\, t^{-(\alpha-1)}\,\ell(t) \quad \text{as} \quad t \to \infty.$$

Finally, $\mathbb{E}[\Lambda^{-(\alpha-1)}] < \infty$ since $-(k(\alpha-1)+\epsilon) < -(\alpha-1) < 0$ if $\alpha \neq 1$ and $\alpha - 1 = 0$ if $\alpha = 1$. □

We pass to the case $\alpha > 2$.

Theorem 4. *Assume that X_1 is of Pareto-type with index $\alpha > 2$ and that $\{N(t);\, t \geq 0\}$ is a mixed Poisson process with mixing random variable Λ. Let $k \in \mathbb{N}$ be fixed. If $k = 1$, assume further that*

$\mathbb{E}[\Lambda^{-(1+\epsilon)}] < \infty$ for some $\epsilon > 0$. If $k \geq 2$, assume further that $\mathbb{E}[\Lambda^{k-2+\epsilon}] < \infty$ and $\mathbb{E}[\Lambda^{-(2k-1+\epsilon)}] < \infty$ for some $\epsilon > 0$. Then for $k < \alpha - 1$:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \left(\frac{\mu_2}{\mu_1^2}\right)^k \mathbb{E}[\Lambda^{-k}]\, t^{-k} \quad as \quad t \to \infty \tag{7}$$

and for $k > \alpha - 1$:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{\alpha}{\mu_1^\alpha} B(2k-\alpha, \alpha)\, \mathbb{E}[\Lambda^{-(\alpha-1)}]\, t^{-(\alpha-1)}\, \ell(t) \quad as \quad t \to \infty. \tag{8}$$

If $k = \alpha - 1$, then:

(i) (7) holds if $\ell(x) = o(1)$ as $x \to \infty$;

(ii) $\mathbb{E}\big[T_{N(t)}^k\big] \sim \left(\left(\frac{\mu_2}{\mu_1^2}\right)^k + C\frac{(k+1)\,B(k-1,k+1)}{\mu_1^{k+1}}\right) \mathbb{E}[\Lambda^{-k}]\, t^{-k}$ as $t \to \infty$ holds if $\lim_{x\to\infty} \ell(x) = C$ for a positive constant C;

(iii) (8) holds otherwise.

Proof. Let $k \in \mathbb{N}$ be fixed. Let $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0^\infty$ and $\alpha > 2$. Since $\mu_1 < \infty$, it follows that $\varphi^{(1)}(0) = -\mu_1$ and $1 - \varphi(s) \sim \mu_1 s$ as $s \downarrow 0$. Hence, $s = 1 - \varphi(\psi(s)) \sim \mu_1\psi(s)$ as $s \downarrow 0$ and we deduce that $\psi(s) \underset{s\downarrow 0}{\sim} s/\mu_1 \in \mathrm{RV}_1^0$, leading to $\lim_{s\downarrow 0} \psi(s) = 0$. For any $n \in \mathbb{N}$, we then have $\varphi^{(2n)} \circ \psi \in \mathrm{RV}_{\alpha-2n}^0$ if $2n > \alpha$ and $\varphi^{(2n)} \circ \psi \in \mathrm{RV}_0^0$ if $2n \leq \alpha$ since $\varphi^{(2n)} \in \mathrm{RV}_{\alpha-2n}^0$ if $2n > \alpha$ and $\varphi^{(2n)} \in \mathrm{RV}_0^0$ if $2n \leq \alpha$ by Lemma 2. Moreover, $-\varphi^{(1)} \circ \psi \in \mathrm{RV}_0^0$ since $-\varphi^{(1)}(s) \underset{s\downarrow 0}{\sim} \mu_1 \in \mathrm{RV}_0^0$.

For simplicity, let us first assume that $\alpha \notin \mathbb{N}$. We start by determining the asymptotic behavior of $B_t(k_1, \ldots, k_r)$ as $t \to \infty$ for any fixed integer $1 \leq r \leq k$. Set $\delta_r := \frac{\zeta}{2k+r}$ with $\zeta = \epsilon$ if $\epsilon \in (0,1)$ or $\zeta \in (0,1)$ otherwise. It follows from Potter's theorem that for this chosen $\delta_r > 0$, there exists $C_r = C_r(\delta_r) > 1$ such that for $t > 0$ and $w > 0$:

$$\begin{aligned} g_t(w; k_1, \ldots, k_r) &\leq C_r\, w^{2k-1}\big(\max\{w^{\delta_r}, w^{-\delta_r}\}\big)^{2k-1}\, w^{r_1\alpha - 2u_1}\big(\max\{w^{\delta_r}, w^{-\delta_r}\}\big)^{r+1}\, q_r(w) \\ &= C_r\, w^{2k-1+r_1\alpha-2u_1} \max\{w^\zeta, w^{-\zeta}\}\, q_r(w) =: h(w) \end{aligned}$$

where r_1 denotes the number of integers among $k_1, \ldots, k_r$ that are greater than $\alpha/2$ and u_1 is the sum of these. It is readily seen that $\int_0^\infty h(w)\, dw < \infty$ if and only if $\int_0^1 w^{2k-1+r_1\alpha-2u_1-\zeta}\, q_r(w)\, dw < \infty$ and $\int_1^\infty w^{2k-1+r_1\alpha-2u_1+\zeta}\, q_r(w)\, dw < \infty$. We have $2 - 2k \leq r_1\alpha - 2u_1 \leq 0$. Indeed, if $r_1 = 0$ then obviously $r_1\alpha - 2u_1 = 0$. Now, if $r_1 \neq 0$ (i.e. $r_1 \geq 1$) then $r_1\alpha - 2u_1 < 0$ on the one hand, and $r_1\alpha - 2u_1 \geq \alpha - 2k$ on the other hand. Consequently, $1 \leq 2k - 1 + r_1\alpha - 2u_1 \leq 2k - 1$. Since $\zeta \in (0,1)$ and $\zeta \leq \epsilon$, we get $0 < r - 2 + \zeta \leq k - 2 + \epsilon$ if $r \geq 2$ and $-(2k - 1 + \epsilon) < -(1 - \zeta) < 0$ if $r = 1$. Moreover, $-(2k - 1 + \epsilon) \leq -(2k - r + \zeta) < 0$. Therefore, using (1) together with Lemma 1 leads to:

$$\int_0^1 w^{2k-1+r_1\alpha-2u_1-\zeta}\, q_r(w)\, dw \leq \int_0^1 w^{1-\zeta}\, q_r(w)\, dw \leq \Gamma(2-\zeta)\, \mathbb{E}[\Lambda^{r-2+\zeta}] < \infty$$

and:

$$\int_1^\infty w^{2k-1+r_1\alpha-2u_1+\zeta}\, q_r(w)\, dw \leq \int_1^\infty w^{2k-1+\zeta}\, q_r(w)\, dw \leq \Gamma(2k+\zeta)\, \mathbb{E}[\Lambda^{-(2k-r+\zeta)}] < \infty.$$

When $k = 1$, obviously $r = 1$ and we get $-(1 + \epsilon) \leq -(1 + \zeta) < -(1 - \zeta) < 0$. The condition $\mathbb{E}[\Lambda^{-(1+\epsilon)}] < \infty$ is thus sufficient for $\mathbb{E}[\Lambda^{-(1-\zeta)}] < \infty$ and $\mathbb{E}[\Lambda^{-(1+\zeta)}] < \infty$ to hold. Hence, the function h is integrable. Finally, $\lim_{t\to\infty} g_t(w; k_1, \ldots, k_r) = w^{2(k-u_1)+r_1\alpha-1} q_r(w)$ for every $w > 0$. Thus, applying Lebesgue's theorem on dominated convergence and using (1), we deduce:

$$\lim_{t\to\infty} \int_0^\infty g_t(w; k_1, \ldots, k_r)\, dw = \int_0^\infty w^{2(k-u_1)+r_1\alpha-1} q_r(w)\, dw = \Gamma(2(k-u_1)+r_1\alpha)\, \mathbb{E}[\Lambda^{r-2(k-u_1)-r_1\alpha}].$$

By virtue of the uniform convergence theorem for slowly varying functions, we get $\ell(1/\psi(s)) \sim \ell(1/s)$ as $s \downarrow 0$. This together with Lemma 2 and $\lim_{s\downarrow 0} \varphi^{(1)}(\psi(s)) = -\mu_1$ yields as $t \to \infty$:

$$\begin{aligned} f_t(k_1, \ldots, k_r) &\sim \frac{\alpha^{r_1} K_1 K_2}{\mu_1}\, t^{r-1}\, \psi^{2(k-u_1)+r_1\alpha-1}(1/t)\, \ell^{r_1}(1/\psi(1/t)) \\ &\sim \frac{\alpha^{r_1} K_1 K_2}{\mu_1^{2(k-u_1)+r_1\alpha}}\, t^{r-2(k-u_1)-r_1\alpha}\, \ell^{r_1}(t) \end{aligned} \tag{9}$$

where $K_1 := \prod_{i\in I_1} \Gamma(2k_i - \alpha)$ with $I_1 := \{i \in \{1,\dots,r\}\colon 2k_i > \alpha\}$ and $K_2 := \prod_{i\in I_2} \mu_{2k_i}$ with $I_2 := \{i \in \{1,\dots,r\}\colon 2k_i < \alpha\}$. Since $\text{card}(I_1) = r_1$ and $\alpha \notin \mathbb{N}$, we have $\text{card}(I_2) = r - r_1$. The asymptotic behavior of $B_t(k_1,\dots,k_r)$ as $t \to \infty$ is therefore given by:

$$B_t(k_1,\dots,k_r) \sim \frac{\alpha^{r_1} K_1 K_2}{\mu_1^{2(k-u_1)+r_1\alpha}}\, \Gamma(2(k-u_1)+r_1\alpha)\, \mathbb{E}[\Lambda^{r-2(k-u_1)-r_1\alpha}]\, t^{r-2(k-u_1)-r_1\alpha}\, \ell^{r_1}(t).$$

It remains to determine the dominating asymptotic term among all possible $B_t(k_1,\dots,k_r)$. For $r_1 > 0$, the largest exponent is obtained with $r_1 = 1$, $u_1 = k$ and thus $r = 1$, so that the asymptotic order is $t^{-(\alpha-1)}\ell(t)$. Note that $r_1 > 0$ is possible for $2k > \alpha$ only. For $r_1 = 0$, obviously $r = k$ (which implies $k_1 = \cdots = k_r = 1$) yields the largest exponent, leading to the asymptotic order t^{-k}. Hence, the asymptotically dominating power among all $B_t(k_1,\dots,k_r)$ is given by $\max\{-(\alpha-1), -k\}$. From this, we see that when $k < \alpha - 1$, the term with $r = k$ dominates and we obtain from (3):

$$\mathbb{E}\big[T^k_{N(t)}\big] \sim \left(\frac{\mu_2}{\mu_1^2}\right)^k \mathbb{E}[\Lambda^{-k}]\, t^{-k} \quad \text{as} \quad t \to \infty.$$

Alternatively, when $k > \alpha - 1$, the term with $r = 1$ dominates and we find:

$$\mathbb{E}\big[T^k_{N(t)}\big] \sim \frac{\alpha}{\mu_1^\alpha}\, B(2k-\alpha, \alpha)\, \mathbb{E}[\Lambda^{-(\alpha-1)}]\, t^{-(\alpha-1)}\, \ell(t) \quad \text{as} \quad t \to \infty$$

which is the same expression as the one obtained in Theorem 3 for $\alpha \in (1,2)$ or $\alpha = 1$ if $\mu_1 < \infty$.

The above conclusions also hold for $\alpha \in \mathbb{N}$ as long as $k \neq \alpha - 1$. Nevertheless, just note that instead of (9), we have the following by virtue of Lemma 2:

$$f_t(k_1,\dots,k_r) \sim \frac{\alpha^{r_1+r_2} K_1 K_3}{\mu_1^{2(k-u_1)+r_1\alpha}}\, t^{r-2(k-u_1)-r_1\alpha}\, \ell^{r_1}(t)\, \tilde{\ell}^{r_2}(t) \quad \text{as} \quad t \to \infty$$

where $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, du \in \text{RV}_0^\infty$, $K_3 := \prod_{i\in I_2\cup I_3} \mu_{2k_i}$ with $I_3 := \{i \in \{1,\dots,r\}\colon 2k_i = \alpha,\, \mu_{2k_i} < \infty\}$ and $r_2 := \text{card}(\{i \in \{1,\dots,r\}\colon 2k_i = \alpha,\, \mu_{2k_i} = \infty\})$. Since $\text{card}(I_2 \cup I_3) = r - r_1 - r_2$, note that $\text{card}(I_2) = r - r_1 - r_2 - \text{card}(I_3)$. When $k = \alpha - 1$, the slowly varying function ℓ determines which of the two terms $t^{-(\alpha-1)}\ell(t)$ (corresponding to $r = 1$) and t^{-k} (corresponding to $r = k$) dominates the asymptotic behavior. If $\ell(x) = o(1)$ as $x \to \infty$, which is in particular fulfilled if $\mu_{k+1} < \infty$, the second term dominates. If $\lim_{x\to\infty} \ell(x) = C$ for a positive constant C, then both terms matter. Otherwise, the first term dominates.

To end the proof, it remains to check that $\mathbb{E}[\Lambda^{-k}] < \infty$ for $k \le \alpha - 1$ and that $\mathbb{E}[\Lambda^{-(\alpha-1)}] < \infty$ for $k > \alpha - 1$. When $k > \alpha - 1$, we have $-(2k-1+\epsilon) < -k < -(\alpha-1) < 0$. When $k \le \alpha - 1$, we have $-(2k-1+\epsilon) < -k < 0$. Thus, we conclude by using Lemma 1. ☐

Finally, we deal with the remaining case $\alpha = 2$.

Theorem 5. *Assume that X_1 is of Pareto-type with index $\alpha = 2$ and that $\{N(t);\, t \ge 0\}$ is a mixed Poisson process with mixing random variable Λ.*

(*i*) *If $\mathbb{E}[\Lambda^{-(1+\epsilon)}] < \infty$ for some $\epsilon > 0$, then:*

$$\mathbb{E}[T_{N(t)}] \sim \begin{cases} \frac{\mu_2\, \mathbb{E}[\Lambda^{-1}]}{\mu_1^2}\frac{1}{t} & \text{if } \mu_2 < \infty \\ \frac{2\, \mathbb{E}[\Lambda^{-1}]}{\mu_1^2}\frac{\tilde{\ell}(t)}{t} & \text{if } \mu_2 = \infty \end{cases} \quad \text{as} \quad t \to \infty$$

where $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, du \in \text{RV}_0^\infty$.

(*ii*) *For any fixed integer $k \ge 2$, if $\mathbb{E}[\Lambda^{k-2+\epsilon}] < \infty$ and $\mathbb{E}[\Lambda^{-(2k-1+\epsilon)}] < \infty$ for some $\epsilon > 0$, then:*

$$\mathbb{E}\big[T^k_{N(t)}\big] \sim \frac{\mathbb{E}[\Lambda^{-1}]}{\mu_1^2\,(k-1)\,(2k-1)}\,\frac{\ell(t)}{t} \quad \text{as} \quad t \to \infty.$$

Proof. Let $k \in \mathbb{N}$ be fixed. Let $1 - F(x) \sim x^{-2}\ell(x)$ as $x \to \infty$ for some $\ell \in \text{RV}_0^\infty$. One can easily verify that Theorem 4 remains true for $\alpha = 2$, except when $k = 1$ if $\mu_2 = \infty$. In the latter case, obviously $r = 1$ and $B_t(k_1,\dots,k_r)$ then becomes:

$$B_t(1) = \int_0^\infty s\, \varphi^{(2)}(s)\, Q_t^{(1)}(\varphi(s))\, ds$$

$$= \underbrace{\frac{-\psi(1/t)}{\varphi^{(1)}(\psi(1/t))}\,\varphi^{(2)}(\psi(1/t))}_{=:f_t(1)} \int_0^\infty \underbrace{\frac{\psi(w/t)}{\psi(1/t)}\,\frac{\varphi^{(2)}(\psi(w/t))}{\varphi^{(2)}(\psi(1/t))}\,\frac{\varphi^{(1)}(\psi(1/t))}{\varphi^{(1)}(\psi(w/t))}\,q_1(w)\,\mathbb{1}_{(0,t)}(w)}_{=:g_t(w;1)}\,dw.$$

Since $\mu_1 < \infty$, it follows that $\varphi^{(1)}(0) = -\mu_1$ and $1-\varphi(s) \sim \mu_1 s$ as $s \downarrow 0$. Hence, $s = 1-\varphi(\psi(s)) \sim \mu_1\psi(s)$ as $s \downarrow 0$ and we deduce that $\psi(s) \underset{s\downarrow 0}{\sim} s/\mu_1 \in \mathrm{RV}_1^0$, leading to $\lim_{s\downarrow 0}\psi(s) = 0$. We then have $\varphi^{(2)} \circ \psi \in \mathrm{RV}_0^0$ since $\varphi^{(2)} \in \mathrm{RV}_0^0$ by Lemma 2. Moreover, $-\varphi^{(1)} \circ \psi \in \mathrm{RV}_0^0$ since $-\varphi^{(1)}(s) \underset{s\downarrow 0}{\sim} \mu_1 \in \mathrm{RV}_0^0$. Set $\delta := \zeta/3$ with $\zeta = \epsilon$ if $\epsilon \in (0,1)$ or $\zeta \in (0,1)$ otherwise. Thus, for this chosen $\delta > 0$, there exists $C = C(\delta) > 1$ such that for $t > 0$ and $w > 0$:

$$g_t(w;1) \le C\,w\left(\max\{w^{\delta}, w^{-\delta}\}\right)^3 q_1(w) = C\,w\,\max\{w^{\zeta}, w^{-\zeta}\}\,q_1(w) =: h(w)$$

by virtue of Potter's theorem. Now, $\int_0^\infty h(w)\,dw < \infty$ if and only if $\int_0^1 w^{1-\zeta}\,q_1(w)\,dw < \infty$ and $\int_1^\infty w^{1+\zeta}\,q_1(w)\,dw < \infty$. Since $\zeta \in (0,1)$ and $\zeta \le \epsilon$, we get $-(1+\epsilon) \le -(1+\zeta) < -(1-\zeta) < 0$. Using (1) together with Lemma 1 then leads to:

$$\int_0^1 w^{1-\zeta}\,q_1(w)\,dw \le \int_0^\infty w^{1-\zeta}\,q_1(w)\,dw = \Gamma(2-\zeta)\,\mathbb{E}[\Lambda^{-(1-\zeta)}] < \infty$$

and:

$$\int_1^\infty w^{1+\zeta}\,q_1(w)\,dw \le \int_0^\infty w^{1+\zeta}\,q_1(w)\,dw = \Gamma(2+\zeta)\,\mathbb{E}[\Lambda^{-(1+\zeta)}] < \infty.$$

Hence, the function h is integrable. Finally, $\lim_{t\to\infty} g_t(w;1) = w\,q_1(w)$ for every $w > 0$. Thus, applying Lebesgue's theorem on dominated convergence and using (1), we deduce:

$$\lim_{t\to\infty}\int_0^\infty g_t(w;1)\,dw = \int_0^\infty w\,q_1(w)\,dw = \mathbb{E}[\Lambda^{-1}].$$

By virtue of the uniform convergence theorem for slowly varying functions, we get $\tilde{\ell}(1/\psi(s)) \sim \tilde{\ell}(1/s)$ as $s \downarrow 0$. This together with $\lim_{s\downarrow 0}\varphi^{(1)}(\psi(s)) = -\mu_1$ and $\varphi^{(2)}(\psi(s)) \sim 2\,\tilde{\ell}(1/\psi(s))$ as $s \downarrow 0$ where $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,du \in \mathrm{RV}_0^\infty$ yields:

$$f_t(1) \sim \frac{2}{\mu_1^2}\frac{\tilde{\ell}(t)}{t} \quad \text{as} \quad t \to \infty.$$

Therefore, we obtain:

$$\mathbb{E}[T_{N(t)}] = B_t(1) \sim \frac{2\,\mathbb{E}[\Lambda^{-1}]}{\mu_1^2}\frac{\tilde{\ell}(t)}{t} \quad \text{as} \quad t \to \infty.$$

Finally, note that $\mathbb{E}[\Lambda^{-1}] < \infty$ since $-(1+\epsilon) < -1$. □

We end by remarking that as in Albrecher and Teugels [2], the integral representation approach that we use in this paper does not permit to get a general asymptotic result for $\mathbb{E}\big[T_{N(t)}^k\big]$ when F belongs to the additive domain of attraction of a normal law, i.e. when F has a slowly varying truncated second moment function. Note however that when F is defined as in Theorem 5, then F belongs to the additive domain of attraction of a normal law.

4. Conclusion

In this paper, we have determined the asymptotic behavior of arbitrary moments of $T_{N(t)}$ as $t \to \infty$ under the conditions that the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$ and that the counting process $\{N(t);\, t \ge 0\}$ is mixed Poisson. Different results have shown up according to the range of α.

In the particular case where the mixing random variable Λ is degenerate at the point 1, our results are then similar to those derived by Albrecher and Teugels [2] where the counting process is non-random.

The *coefficient of variation* of a positive random variable X is defined by:

$$\mathrm{CoVar}(X) := \frac{\sqrt{\mathbb{V}[X]}}{\mathbb{E}[X]}$$

where $\mathbb{V}[X]$ denotes the variance of X. This risk measure is frequently used in practice. For instance, it is very popular among actuaries. From a random sample $X_1, \ldots, X_{N(t)}$ from X of random size $N(t)$ from a nonnegative integer-valued distribution, the coefficient of variation $\mathrm{CoVar}(X)$ is naturally estimated by the sample coefficient of variation of X defined by:

$$\widehat{\mathrm{CoVar}(X)} := \frac{S}{\overline{X}}$$

where $\overline{X} := \frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i$ is the sample mean and $S^2 := \frac{1}{N(t)} \sum_{i=1}^{N(t)} \left(X_i - \overline{X}\right)^2$ is the sample variance. The properties of the sample coefficient of variation $\widehat{\mathrm{CoVar}(X)}$ are usually studied under the tacite assumption of the finiteness of sufficiently many moments of X. However, the existence of moments of X is not always guaranteed in practical applications. It is therefore useful to investigate asymptotic properties of $\widehat{\mathrm{CoVar}(X)}$ also when such a moment condition is not satisfied. It turns out that this can be achieved by using results on $T_{N(t)}$ due to the following equality:

$$\widehat{\mathrm{CoVar}(X)} = \sqrt{N(t)\, T_{N(t)} - 1}. \tag{10}$$

Ladoucette and Teugels [8] focus on *weak convergence* by deriving limit distributions for the appropriately normalized ratio $T_{N(t)}$ as $t \to \infty$ when X is of Pareto-type with index $\alpha > 0$ and the counting process $\{N(t);\, t \geq 0\}$ satisfies some convergence conditions according to the range of α. Armed with their results on $T_{N(t)}$ and thanks to the relation (10), they also derive asymptotic properties of the sample coefficient of variation, even when the first moment and/or the second moment of X do not exist. Furthermore, Ladoucette and Teugels [8] adapt the methodology to derive asymptotic properties of another measure of variation, namely the *sample dispersion*. Recall that the value of the *dispersion* allows to compare the volatility with respect to the Poisson case.

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