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LIMIT DISTRIBUTIONS FOR THE RATIO OF THE RANDOM SUM OF SQUARES TO THE SQUARE OF THE RANDOM SUM WITH APPLICATIONS TO RISK MEASURES

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Abstract. Let $\{X_1, X_2, \ldots\}$ be a sequence of independent and identically distributed positive random variables of Pareto-type and let $\{N(t);\, t \geqslant 0\}$ be a counting process independent of the X_i's. For any fixed $t \geqslant 0$, define:

$$T_{N(t)} := \frac{X_1^2 + X_2^2 + \cdots + X_{N(t)}^2}{(X_1 + X_2 + \cdots + X_{N(t)})^2}$$

if $N(t) \geqslant 1$ and $T_{N(t)} := 0$ otherwise. We derive limits in distribution for $T_{N(t)}$ under some convergence conditions on the counting process. This is even achieved when both the numerator and the denominator defining $T_{N(t)}$ exhibit an erratic behavior ($\mathbb{E}X_1 = \infty$) or when only the numerator has an erratic behavior ($\mathbb{E}X_1 < \infty$ and $\mathbb{E}X_1^2 = \infty$). Armed with these results, we obtain asymptotic properties of two popular risk measures, namely the sample coefficient of variation and the sample dispersion.

1. Introduction

Let $\{X_1, X_2, \ldots\}$ be a sequence of independent and identically distributed positive random variables with distribution function F and let $\{N(t);\, t \geqslant 0\}$ be a counting process independent of the X_i's. For any fixed $t \geqslant 0$, define the random variable $T_{N(t)}$ by:

$$T_{N(t)} := \frac{X_1^2 + X_2^2 + \cdots + X_{N(t)}^2}{\bigl(X_1 + X_2 + \cdots + X_{N(t)}\bigr)^2} \tag{1.1}$$

if $N(t) \geqslant 1$ and $T_{N(t)} := 0$ otherwise.

The limiting behavior of arbitrary moments of the ratio $T_{N(t)}$ is derived in Ladoucette [**8**] under the conditions that the distribution function F of X_1 is of *Pareto-type* with index $\alpha > 0$ and that the counting process $\{N(t);\, t \geqslant 0\}$ is

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mixed Poisson. In this paper, we focus on *weak convergence* in deriving limits in distribution for the appropriately normalized $T_{N(t)}$. We still assume that F is of Pareto-type with index $\alpha > 0$ except in one result where we assume that the fourth moment of X_1 exists. Our results are derived under the additional condition that the counting process either *$\mathcal{D}$-averages in time* or *p-averages in time* according to the range of α. We therefore generalize results established by Albrecher et al. [**1**] where the counting process is non-random (deterministic case). The appropriate definitions along with some properties are given in Section 2.

The results of the paper rely on the theory of functions of *regular variation* (e.g., Bingham et al. [**4**]). Recall that a distribution function F on $(0,\infty)$ of Pareto-type with index $\alpha > 0$ is defined by:

$$1 - F(x) \sim x^{-\alpha}\ell(x) \quad \text{as} \quad x \to \infty \tag{1.2}$$

for a slowly varying function ℓ, and therefore has a regularly varying tail $1-F$ with index $-\alpha < 0$.

Let μ_β denote the moment of order $\beta > 0$ of X_1, i.e.:

$$\mu_\beta := \mathbb{E}X_1^\beta = \beta \int_0^\infty x^{\beta-1}\left(1 - F(x)\right) dx \leqslant \infty.$$

Clearly, both the numerator and the denominator defining $T_{N(t)}$ exhibit an erratic behavior if $\mu_1 = \infty$, whereas this is the case only for the numerator if $\mu_1 < \infty$ and $\mu_2 = \infty$. When X_1 (or equivalently F) is of Pareto-type with index $\alpha > 0$, it turns out that μ_β is finite if $\beta < \alpha$ but infinite whenever $\beta > \alpha$. In particular, $\mu_1 < \infty$ if $\alpha > 1$ while $\mu_2 < \infty$ as soon as $\alpha > 2$. Since the asymptotic behavior of $T_{N(t)}$ is influenced by the finiteness of μ_1 and/or μ_2, different limiting distributions will consequently show up according to the range of α. This is expressed in our main results given in Section 3. In Section 4, we use our results to study the asymptotic behavior of the *sample coefficient of variation* and the *sample dispersion* through limits in distribution.

The *coefficient of variation* of a positive random variable X is defined by:

$$\mathrm{CoVar}(X) := \frac{\sqrt{\mathbb{V}X}}{\mathbb{E}X}$$

where $\mathbb{V}X$ denotes the variance of X. This risk measure is frequently used in practice and is very popular among actuaries. From a random sample $X_1, \ldots, X_{N(t)}$ from X of random size $N(t)$ from a nonnegative integer-valued distribution, the coefficient of variation $\mathrm{CoVar}(X)$ is naturally estimated by the sample coefficient of variation defined by:

$$\widehat{\mathrm{CoVar}(X)} := \frac{S}{\overline{X}} \tag{1.3}$$

where $\overline{X} := \frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i$ is the sample mean and $S^2 := \frac{1}{N(t)} \sum_{i=1}^{N(t)} \left(X_i - \overline{X}\right)^2$ is the sample variance. The properties of the sample coefficient of variation are usually studied under the tacite assumption of the finiteness of sufficiently many moments of X. However, the existence of moments of X is not always guaranteed in practical applications. It is therefore useful to investigate the limiting behavior of $\widehat{\mathrm{CoVar}(X)}$ also in these cases. It turns out that this can be achieved by using results on $T_{N(t)}$. Indeed, the quantity $T_{N(t)}$ appears as a basic ingredient in the study of the sample coefficient of variation due to:

$$\widehat{\mathrm{CoVar}(X)} = \sqrt{N(t)\,T_{N(t)} - 1}. \tag{1.4}$$

In Subsection 4.1, we take advantage from this link to derive asymptotic properties of the sample coefficient of variation under the same assumptions on X and on the counting process as in Section 3. Note that this is done even when the first moment and/or the second moment of X do not exist.

Another risk measure of the positive random variable X that is very popular is the *dispersion* defined by:

$$\mathrm{D}(X) := \frac{\mathbb{V}X}{\mathbb{E}X}.$$

For instance, in a (re)insurance context, the value of the dispersion is used to compare the volatility of a portfolio with respect to the Poisson case for which the dispersion equals 1. Similarly to the coefficient of variation, the dispersion $\mathrm{D}(X)$ is typically estimated by the sample dispersion defined by:

$$\widehat{\mathrm{D}(X)} := \frac{S^2}{\overline{X}}. \tag{1.5}$$

Defining the random variable $C_{N(t)}$ for any fixed $t \geqslant 0$ by:

$$C_{N(t)} := \frac{X_1^2 + X_2^2 + \cdots + X_{N(t)}^2}{X_1 + X_2 + \cdots + X_{N(t)}} \tag{1.6}$$

if $N(t) \geqslant 1$ and $C_{N(t)} := 0$ otherwise, leads to the following link with the sample dispersion:

$$\widehat{\mathrm{D}(X)} = C_{N(t)} - \overline{X}. \tag{1.7}$$

It turns out that results from Section 3 can be used to derive asymptotic properties of the sample dispersion from those of $C_{N(t)}$. The results are given in Subsection 4.2 under the same conditions on X and on the counting process as in Section 3. As for the sample coefficient of variation, cases where the first moments of X do not exist are also considered.

2. Preliminaries

Though standard notations, we mention that $\xrightarrow{a.s.}$, $\xrightarrow{p}$, $\xrightarrow{\mathcal{D}}$ stand for convergence almost surely, in probability and in distribution, respectively. Equality in distribution is denoted by $\overset{\mathcal{D}}{=}$. For two measurable functions f and g, we write $f(x) = o(g(x))$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 0$ and $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. Finally, $\Gamma(\cdot)$ denotes the gamma function.

Let $\{N(t);\, t \geqslant 0\}$ be a counting process. For any fixed $t \geqslant 0$, the probability generating function of the random variable $N(t)$ is defined by:

$$Q_t(z) := \mathbb{E}\{z^{N(t)}\} = \sum_{n=0}^{\infty} \mathbb{P}[N(t) = n]\, z^n, \ \ |z| \leqslant 1.$$

Most of our results are obtained by assuming that the counting process satisfies the following condition:

$$\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda \quad \text{as} \quad t \to \infty$$

where the limiting random variable Λ is such that $\mathbb{P}[\Lambda > 0] = 1$. The counting process is then said to $\mathcal{D}$-average in time to Λ. In two cases however, we will need to require the stronger condition that the above convergence holds in probability rather than in distribution, i.e.:

$$\frac{N(t)}{t} \xrightarrow{p} \Lambda \quad \text{as} \quad t \to \infty$$

in which case the counting process is said to p-average in time to Λ. Whether the counting process $\mathcal{D}$-averages in time or p-averages in time, we have $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$. Very popular counting processes $\mathcal{D}$-average in time. The deterministic case provides a first example for which Λ is degenerate at the point 1. Any mixed Poisson process obviously $\mathcal{D}$-averages in time to its mixing random variable. We refer to the monograph by Grandell [**7**] for a very thorough treatment of mixed Poisson processes and their properties. Finally, any renewal process generated by a positive distribution with finite mean μ also $\mathcal{D}$-averages in time with Λ degenerate at the point $1/\mu$.

The convergence in distribution being equivalent to the pointwise convergence of the corresponding Laplace transforms, a counting process that $\mathcal{D}$-averages in time or p-averages in time to Λ satisfies:

$$\lim_{t\to\infty} \mathbb{E}\Big\{e^{-\theta N(t)/t}\Big\} = \mathbb{E}\big\{e^{-\theta\Lambda}\big\}, \ \theta \geqslant 0. \tag{2.1}$$

For every $\theta \geqslant 0$, define $u_\theta(x) := e^{-\theta x}$ for $x \geqslant 0$. The family of functions $\{u_\theta\}_{\theta\geqslant 0}$ being equicontinuous provided θ is restricted to a finite interval, the convergence in (2.1) holds uniformly in every finite θ-interval (e.g., Corollary page 252 of Feller [**6**]).

As specified above, most of our results are derived under the condition that the tail of F satisfies (1.2), i.e. that $1 - F$ is regularly varying with index $-\alpha < 0$. Recall that a measurable function $f\colon (0,\infty) \to (0,\infty)$ is regularly varying with index $\gamma \in \mathbb{R}$ (written $f \in \mathrm{RV}_\gamma$) if for all $x > 0$, $\lim_{t\to\infty} f(tx)/f(t) = x^\gamma$. When $\gamma = 0$, the function f is said to be slowly varying. For a textbook treatment on the theory of functions of regular variation, we refer to Bingham et al. [**4**]. It is well-known that the tail condition (1.2) appears as the essential condition in the Fréchet-Pareto domain of attraction problem of extreme value theory. For a recent treatment, see Beirlant et al. [**2**]. When $\alpha \in (0, 2)$, the condition is also necessary and sufficient for F to belong to the additive domain of attraction of a non-normal α-stable distribution (e.g., Theorem 8.3.1 of Bingham et al. [**4**]). Recall that a stable random variable X is positive if and only if $X \overset{\mathcal{D}}{=} cU_\gamma$ for some $c > 0$ and $\gamma \in (0, 1)$, where the random variable U_γ has the following Laplace transform:

$$\mathbb{E}\big\{e^{-\theta U_\gamma}\big\} = e^{-\theta^\gamma \Gamma(1-\gamma)}, \ \theta \geqslant 0. \tag{2.2}$$

Any random variable U_γ having the Laplace transform (2.2) with $\gamma \in (0,1)$ is then positive γ-stable.

Finally, we give a general result that will prove to be very useful later on.

LEMMA 2.1. *Let* $\{Y_n;\, n \geqslant 1\}$ *be a general sequence of random variables and* $\{M(t);\, t \geqslant 0\}$ *be a process of nonnegative integer-valued random variables. Assume that* $\{Y_n;\, n \geqslant 1\}$ *and* $\{M(t);\, t \geqslant 0\}$ *are independent and that* $M(t) \xrightarrow{p} \infty$ *as* $t \to \infty$*. If* $Y_n \xrightarrow{\mathcal{D}} Y$ *as* $n \to \infty$ *then* $Y_{M(t)} \xrightarrow{\mathcal{D}} Y$ *as* $t \to \infty$*.*

PROOF. Let y be a continuity point of the distribution function F_Y of Y. For every $\epsilon \in (0,1)$, there exists $n_0 = n_0(\epsilon, y) \in \mathbb{N}$ such that $|\mathbb{P}[Y_n \leqslant y] - F_Y(y)| \leqslant \epsilon$ for all $n > n_0$, since $Y_n \xrightarrow{\mathcal{D}} Y$ as $n \to \infty$. By using conditioning and independence arguments, we then obtain:

$$\begin{aligned}
\left|\mathbb{P}[Y_{M(t)} \leqslant y] - F_Y(y)\right| &= \left|\left(\sum_{n=0}^{n_0} + \sum_{n=n_0+1}^{\infty}\right)\{\mathbb{P}[Y_n \leqslant y] - F_Y(y)\}\,\mathbb{P}[M(t) = n]\right| \\
&\leqslant \sum_{n=0}^{n_0} |\mathbb{P}[Y_n \leqslant y] - F_Y(y)|\,\mathbb{P}[M(t) = n] \\
&\quad + \sum_{n=n_0+1}^{\infty} |\mathbb{P}[Y_n \leqslant y] - F_Y(y)|\,\mathbb{P}[M(t) = n] \\
&\leqslant \mathbb{P}[M(t) \leqslant n_0] + \epsilon\,\mathbb{P}[M(t) > n_0].
\end{aligned}$$

Since $M(t) \xrightarrow{p} \infty$ as $t \to \infty$, it follows that $\limsup_{t\to\infty} \left|\mathbb{P}[Y_{M(t)} \leqslant y] - F_Y(y)\right| \leqslant \epsilon$. The claim is proved upon letting $\epsilon \downarrow 0$. □

3. Convergence in Distribution for $T_{N(t)}$

We derive asymptotic distributions for the properly normalized ratio $T_{N(t)}$ defined in (1.1) under the condition that the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$ as defined in (1.2). The last result is even established by assuming that $\mu_4 < \infty$ and consequently holds in the cases $\alpha = 4$ if $\mu_4 < \infty$ and $\alpha > 4$. Throughout the section, the counting process $\{N(t);\, t \geqslant 0\}$ is assumed to $\mathcal{D}$-average in time except for two results where we need to make the stronger assumption that it p-averages in time.

THEOREM 3.1. *Assume that* X_1 *is of Pareto-type with index* $\alpha \in (0,1)$ *and that* $\{N(t);\, t \geqslant 0\}$ $\mathcal{D}$*-averages in time to the random variable* Λ*. Then:*

$$T_{N(t)} \xrightarrow{\mathcal{D}} \frac{U_{\alpha/2}}{U_\alpha^2} \quad \text{as} \quad t \to \infty$$

where the random vector $(U_{\alpha/2}, U_\alpha)'$ *has the Laplace transform:*

$$\mathbb{E}\{e^{-rU_{\alpha/2} - sU_\alpha}\} = \exp\left(-\int_0^\infty e^{-ru^2 - su}\,(2ru + s)\,u^{-\alpha}\,du\right), \; r \geqslant 0,\, s \geqslant 0. \tag{3.1}$$

In particular, the marginal random variables $U_{\alpha/2}$ *and* U_α *are positive stable with respective exponent* $\alpha/2$ *and* α *and have the Laplace transform* (2.2) *with* $\gamma = \alpha/2$ *and* $\gamma = \alpha$ *respectively.*

Proof. Let $1-F(x)\sim x^{-\alpha}\ell(x)$ as $x\to\infty$ for some $\ell\in \mathrm{RV}_0$ and $\alpha\in(0,1)$. Define a sequence $(a_t)_{t>0}$ by $1-F(a_t)\sim 1/t$ as $t\to\infty$, i.e. $\lim_{t\to\infty} t\,a_t^{-\alpha}\ell(a_t)=1$. Notice that $a_t\in \mathrm{RV}_{1/\alpha}$.

Let $r\geqslant 0$ and $s\geqslant 0$ be fixed. By using conditioning and independence arguments, we obtain:

$$\mathbb{E}\Big\{\exp\Big(-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)}X_i^2-s\frac{1}{a_t}\sum_{i=1}^{N(t)}X_i\Big)\Big\}=Q_t\Big(e^{-\delta_{\alpha,t}(r,s)/t}\Big)$$

with $\delta_{\alpha,t}(r,s):=-t\,\log\int_0^\infty e^{-r(x/a_t)^2-sx/a_t}\,dF(x)\in[0,\infty)$. We know from the proof of Theorem 2.1 of Albrecher et al. [**1**] that:

$$\lim_{t\to\infty}\delta_{\alpha,t}(r,s)=\int_0^\infty e^{-ru^2-su}\,(2ru+s)\,u^{-\alpha}\,du=:\delta_\alpha(r,s)\in[0,\infty)$$

and that:

$$\left(\frac{1}{a_n^2}\sum_{i=1}^{n}X_i^2,\ \frac{1}{a_n}\sum_{i=1}^{n}X_i\right)'\xrightarrow{\mathcal{D}}(U_{\alpha/2},U_\alpha)'\quad\text{as}\quad n\to\infty$$

where the Laplace transform of $(U_{\alpha/2},U_\alpha)'$ is given by:

$$\mathbb{E}\big\{e^{-rU_{\alpha/2}-sU_\alpha}\big\}=e^{-\delta_\alpha(r,s)},\ r\geqslant 0,\ s\geqslant 0.$$

It follows in particular that $U_{\alpha/2}$ and U_α each have the Laplace transform (2.2) with $\gamma=\alpha/2$ for the former and $\gamma=\alpha$ for the latter, meaning that $U_{\alpha/2}$ is positive $\alpha/2$-stable and that U_α is positive α-stable.

Define $\varphi_t(\theta):=Q_t\big(e^{-\theta/t}\big)=\mathbb{E}\big\{e^{-\theta N(t)/t}\big\}$ for $\theta\geqslant 0$ so that $\lim_{t\to\infty}\varphi_t(\theta)=\mathbb{E}\big\{e^{-\theta\Lambda}\big\}=:\varphi(\theta)$ by (2.1). Write the following triangular inequality:

$$|\varphi_t(\delta_{\alpha,t}(r,s))-\varphi(\delta_\alpha(r,s))|\leqslant|\varphi_t(\delta_{\alpha,t}(r,s))-\varphi(\delta_{\alpha,t}(r,s))|+|\varphi(\delta_{\alpha,t}(r,s))-\varphi(\delta_\alpha(r,s))|\,.$$

On the one hand, $\lim_{t\to\infty}|\varphi(\delta_{\alpha,t}(r,s))-\varphi(\delta_\alpha(r,s))|=0$ by continuity of φ. On the other hand, for t large enough, there exist reals a,b with $0\leqslant a\leqslant\delta_\alpha(r,s)<b$ such that $\delta_{\alpha,t}(r,s)\in[a,b]$. Then, $\lim_{t\to\infty}|\varphi_t(\delta_{\alpha,t}(r,s))-\varphi(\delta_{\alpha,t}(r,s))|=0$ if and only if $\lim_{t\to\infty}\sup_{\theta\in[a,b]}|\varphi_t(\theta)-\varphi(\theta)|=0$. The latter is true since (2.1) holds uniformly in every finite θ-interval. As a consequence, we have $\lim_{t\to\infty}\varphi_t(\delta_{\alpha,t}(r,s))=\varphi(\delta_\alpha(r,s))$, that is:

$$\lim_{t\to\infty}\mathbb{E}\Big\{\exp\Big(-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)}X_i^2-s\frac{1}{a_t}\sum_{i=1}^{N(t)}X_i\Big)\Big\}=\mathbb{E}\Big\{e^{-\delta_\alpha(r,s)\,\Lambda}\Big\},\ r\geqslant 0,\ s\geqslant 0.$$

However, since Λ is independent of $U_{\alpha/2}$ and U_α, we readily compute by using conditioning arguments:

$$\mathbb{E}\Big\{e^{-rU_{\alpha/2}\Lambda^{2/\alpha}-sU_\alpha\Lambda^{1/\alpha}}\Big\}=\mathbb{E}\Big\{e^{-\delta_\alpha(r,s)\,\Lambda}\Big\},\ r\geqslant 0,\ s\geqslant 0. \tag{3.2}$$

Hence, we have proved the following:

$$\left(\frac{1}{a_t^2}\sum_{i=1}^{N(t)}X_i^2,\ \frac{1}{a_t}\sum_{i=1}^{N(t)}X_i\right)'\xrightarrow{\mathcal{D}}(U_{\alpha/2}\,\Lambda^{2/\alpha},U_\alpha\,\Lambda^{1/\alpha})'\quad\text{as}\quad t\to\infty \tag{3.3}$$

where $(U_{\alpha/2}\,\Lambda^{2/\alpha}, U_\alpha\,\Lambda^{1/\alpha})'$ has the Laplace transform (3.2). The Continuous Mapping Theorem (CMT), see e.g. Corollary 1 page 31 of Billingsley [**3**], finally gives:

$$T_{N(t)} = \left(\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2\right)\left(\frac{1}{a_t}\sum_{i=1}^{N(t)} X_i\right)^{-2} \xrightarrow{\mathcal{D}} \frac{U_{\alpha/2}}{U_\alpha^2} \quad \text{as} \quad t\to\infty$$

where $(U_{\alpha/2}, U_\alpha)'$ has the Laplace transform (3.1). This concludes the proof. □

THEOREM 3.2. *Assume that X_1 is of Pareto-type with index $\alpha = 1$ and $\mu_1 = \infty$. Assume that $\{N(t);\, t \geqslant 0\}$ $\mathcal{D}$-averages in time to the random variable Λ. Then:*

$$\left(\frac{a'_t}{a_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} U_{1/2} \quad \textit{as} \quad t\to\infty$$

where $U_{1/2}$ is a positive 1/2-stable random variable with Laplace transform (2.2) for $\gamma = 1/2$ and where the sequences $(a_t)_{t>0}$ and $(a'_t)_{t>0}$ are respectively defined by $\lim_{t\to\infty} t\, a_t^{-1}\ell(a_t) = 1$ and $\lim_{t\to\infty} t\, a_t'^{-1}\tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, du \in \mathrm{RV}_0$.

PROOF. Let $1-F(x) \sim x^{-1}\ell(x)$ as $x\to\infty$ for some $\ell \in \mathrm{RV}_0$ such that $\mu_1 = \infty$. Define a sequence $(a_t)_{t>0}$ by $1-F(a_t) \sim 1/t$ as $t\to\infty$, i.e. $\lim_{t\to\infty} t\, a_t^{-1}\ell(a_t) = 1$, and a sequence $(a'_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t'^{-1}\tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, du$. Note that $\tilde{\ell} \in \mathrm{RV}_0$ and $\lim_{x\to\infty} \ell(x)/\tilde{\ell}(x) = 0$ (e.g., Proposition 1.5.9a of Bingham et al. [**4**]).

Let $r \geqslant 0$ and $s \geqslant 0$ be fixed. We readily compute:

$$\mathbb{E}\left\{\exp\left(-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i\right)\right\} = Q_t\left(e^{-\delta_t(r,s)/t}\right)$$

with $\delta_t(r,s) := -t\log\int_0^\infty e^{-r(x/a_t)^2 - sx/a'_t}\, dF(x) \in [0,\infty)$. The Dominated Convergence Theorem (DCT) gives $\lim_{t\to\infty}\int_0^\infty e^{-r(x/a_t)^2 - sx/a'_t}\, dF(x) = 1$, so that:

$$\begin{aligned}\delta_t(r,s) \underset{t\uparrow\infty}{\sim} \;& 2r\int_0^\infty y\, e^{-ry^2 - sya_t/a'_t}\, t\,(1-F(a_t y))\, dy \\ & + \frac{s^2 t}{a'_t}\int_0^\infty e^{-sy}\int_0^{a'_t y} (1-F(x))\, e^{-r(x/a_t)^2}\, dx\, dy.\end{aligned}$$

Since $\lim_{x\to\infty} \ell(x)/\tilde{\ell}(x) = 0$, we obtain with de Bruijn conjugate arguments that $\lim_{t\to\infty} a_t/a'_t = 0$. Note however that $a_t/a'_t \in \mathrm{RV}_0$ since $a_t \in \mathrm{RV}_1$ and $a'_t \in \mathrm{RV}_1$. Applying Potter's theorem (e.g., Theorem 1.5.6 of Bingham et al. [**4**]) and the DCT then leads to:

$$\lim_{t\to\infty} 2r\int_0^\infty y\, e^{-ry^2 - sya_t/a'_t}\, t\,(1-F(a_t y))\, dy = 2r\int_0^\infty e^{-ry^2}\, dy = \sqrt{r\pi}.$$

Since $\mu_1 = \infty$, we have $\tilde{\ell}(x) \sim \int_0^x (1-F(u))\, du$ as $x\to\infty$. For any $y > 0$, we then obtain as $t\to\infty$:

$$\int_0^{a'_t y} (1-F(x))\, e^{-r(x/a_t)^2}\, dx \sim \int_0^{a'_t y} (1-F(x))\, dx \sim \tilde{\ell}(a'_t y) \sim \tilde{\ell}(a'_t) \sim \frac{a'_t}{t}$$

so that the DCT leads to:

$$\lim_{t\to\infty} \frac{s^2 t}{a'_t} \int_0^\infty e^{-sy} \int_0^{a'_t y} (1 - F(x))\, e^{-r(x/a_t)^2} dx\, dy = s.$$

It follows that $\lim_{t\to\infty} \delta_t(r,s) = \sqrt{r\pi} + s$. A similar argument as in the proof of Theorem 3.1 applied to the convergence of $Q_t\big(e^{-\delta_t(r,s)/t}\big)$ then yields:

$$\lim_{t\to\infty} \mathbb{E}\bigg\{\exp\bigg(-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i\bigg)\bigg\} = \mathbb{E}\Big\{e^{-\sqrt{r\pi}\,\Lambda - s\Lambda}\Big\},\ r \geqslant 0,\ s \geqslant 0.$$

Repeating the proof of Theorem 3.1 with $s = 0$ shows that $a_t^{-2}\sum_{i=1}^{N(t)} X_i^2 \xrightarrow{\mathcal{D}} U_{1/2}\,\Lambda^2$ as $t \to \infty$, where $U_{1/2}$ is a positive 1/2-stable random variable independent of Λ with Laplace transform (2.2) for $\gamma = 1/2$. From this independence, we get by using conditioning arguments:

$$\mathbb{E}\Big\{e^{-rU_{1/2}\,\Lambda^2 - s\Lambda}\Big\} = \mathbb{E}\Big\{e^{-\sqrt{r\pi}\,\Lambda - s\Lambda}\Big\}, \quad r \geqslant 0,\ s \geqslant 0. \tag{3.4}$$

It then follows that:

$$\bigg(\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2,\ \frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i\bigg)' \xrightarrow{\mathcal{D}} (U_{1/2}\,\Lambda^2, \Lambda)' \quad \text{as} \quad t \to \infty \tag{3.5}$$

where $(U_{1/2}\,\Lambda^2, \Lambda)'$ has the Laplace transform (3.4). The CMT finally gives:

$$\bigg(\frac{a'_t}{a_t}\bigg)^2 T_{N(t)} = \bigg(\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2\bigg)\bigg(\frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i\bigg)^{-2} \xrightarrow{\mathcal{D}} U_{1/2} \quad \text{as} \quad t \to \infty$$

and the proof is complete. □

THEOREM 3.3. *Assume that* X_1 *is of Pareto-type with index* $\alpha \in (1,2)$ *(including* $\alpha = 1$ *if* $\mu_1 < \infty$*) and that* $\{N(t);\, t \geqslant 0\}$ $\mathcal{D}$*-averages in time to the random variable* Λ*.*

(*a*) *Then:* $\Big(\dfrac{N(t)}{a_t}\Big)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \dfrac{1}{\mu_1^2}\, U_{\alpha/2}\,\Lambda^{2/\alpha}$ *as* $t \to \infty$.

(*b*) *Then:* $\Big(\dfrac{t}{a_t}\Big)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \dfrac{1}{\mu_1^2}\,\dfrac{U_{\alpha/2}}{\Lambda^{2-2/\alpha}}$ *as* $t \to \infty$.

In (a) and (b), $U_{\alpha/2}$ *is a positive* $\alpha/2$*-stable random variable independent of* Λ *with Laplace transform (2.2) for* $\gamma = \alpha/2$*. Moreover, the sequence* $(a_t)_{t>0}$ *is defined by* $\lim_{t\to\infty} t\, a_t^{-\alpha}\ell(a_t) = 1$*.*

PROOF. Let $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$ and $\alpha \in (1,2)$ or $\alpha = 1$ if $\mu_1 < \infty$. Define a sequence $(a_t)_{t>0}$ by $1 - F(a_t) \sim 1/t$ as $t \to \infty$, i.e. $\lim_{t\to\infty} t\, a_t^{-\alpha}\ell(a_t) = 1$. Note that $a_t \in \mathrm{RV}_{1/\alpha}$.

(*a*) Since $\mu_1 < \infty$ and $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, we get $\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i \xrightarrow{p} \mu_1$ as $t \to \infty$ by Lemma 2.1. Repeating the proof of Theorem 3.1 with $s = 0$ shows that $a_t^{-2}\sum_{i=1}^{N(t)} X_i^2 \xrightarrow{\mathcal{D}} U_{\alpha/2}\,\Lambda^{2/\alpha}$ as $t \to \infty$, where $U_{\alpha/2}$ is a positive $\alpha/2$-stable random variable independent of Λ with Laplace transform (2.2) for $\gamma = \alpha/2$. Slutsky's theorem (e.g., Corollary page 97 of Chung [**5**]) and the CMT then yield:

$$\left(\frac{N(t)}{a_t}\right)^2 T_{N(t)} = \left(\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2\right)\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i\right)^{-2} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\, U_{\alpha/2}\, \Lambda^{2/\alpha} \quad \text{as} \quad t \to \infty.$$

(*b*) Let $r \geqslant 0$ and $s \geqslant 0$ be fixed. We readily compute:

$$\mathbb{E}\left\{\exp\left(-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{t}\sum_{i=1}^{N(t)} X_i\right)\right\} = Q_t\left(e^{-\delta_{\alpha,t}(r,s)/t}\right)$$

with $\delta_{\alpha,t}(r,s) := -t\,\log\int_0^\infty e^{-r(x/a_t)^2 - sx/t}\,dF(x) \in [0,\infty)$. By virtue of the DCT, we have $\lim_{t\to\infty}\int_0^\infty e^{-r(x/a_t)^2-sx/t}\,dF(x) = 1$. It then follows that:

$$\delta_{\alpha,t}(r,s) \underset{t\uparrow\infty}{\sim} 2r\int_0^\infty y\,e^{-ry^2-sya_t/t}\,t\,(1-F(a_t y))\,dy$$
$$+ s\int_0^\infty (1-F(x))\,e^{-r(x/a_t)^2-sx/t}\,dx.$$

If $\alpha = 1$, we have $\ell(x) = o(1)$ as $x \to \infty$ since $\mu_1 < \infty$ so that $a_t/t \sim \ell(a_t) \to 0$ as $t \to \infty$. If $\alpha \in (1,2)$, we have $a_t/t \sim a_t^{1-\alpha}\ell(a_t) \to 0$ as $t \to \infty$ since $1-\alpha \in (-1,0)$. In both cases, we obtain that $\lim_{t\to\infty} a_t/t = 0$. Applying Potter's theorem and the DCT then leads to:

$$\lim_{t\to\infty} 2r\int_0^\infty ye^{-ry^2-sya_t/t}t\,(1-F(a_t y))\,dy = 2r\int_0^\infty y^{1-\alpha}e^{-ry^2}\,dy = r^{\alpha/2}\Gamma(1-\alpha/2).$$

Since $\mu_1 < \infty$, an application of the DCT gives:

$$\lim_{t\to\infty} s\int_0^\infty (1-F(x))\,e^{-r(x/a_t)^2-sx/t}\,dx = s\mu_1.$$

It follows that $\lim_{t\to\infty}\delta_{\alpha,t}(r,s) = r^{\alpha/2}\Gamma(1-\alpha/2) + s\mu_1$. A similar argument as in the proof of Theorem 3.1 applied to the convergence of $Q_t\left(e^{-\delta_{\alpha,t}(r,s)/t}\right)$ then yields:

$$\lim_{t\to\infty}\mathbb{E}\left\{\exp\left(-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{t}\sum_{i=1}^{N(t)} X_i\right)\right\} = \mathbb{E}\left\{e^{-r^{\alpha/2}\Gamma(1-\alpha/2)\,\Lambda - s\mu_1\Lambda}\right\},\ \begin{matrix} r \geqslant 0, \\ s \geqslant 0. \end{matrix}$$

We know from (*a*) that $a_t^{-2}\sum_{i=1}^{N(t)} X_i^2 \xrightarrow{\mathcal{D}} U_{\alpha/2}\,\Lambda^{2/\alpha}$ as $t \to \infty$, where $U_{\alpha/2}$ is a positive $\alpha/2$-stable random variable independent of Λ with Laplace transform (2.2) for $\gamma = \alpha/2$. From this independence, we get by using conditioning arguments:

$$\mathbb{E}\left\{e^{-rU_{\alpha/2}\,\Lambda^{2/\alpha} - s\mu_1\Lambda}\right\} = \mathbb{E}\left\{e^{-r^{\alpha/2}\Gamma(1-\alpha/2)\,\Lambda - s\mu_1\Lambda}\right\},\ r \geqslant 0,\ s \geqslant 0. \tag{3.6}$$

It then follows that:

$$\left(\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2,\ \frac{1}{t}\sum_{i=1}^{N(t)} X_i\right)' \xrightarrow{\mathcal{D}} (U_{\alpha/2}\,\Lambda^{2/\alpha}, \mu_1\Lambda)' \quad \text{as} \quad t \to \infty \tag{3.7}$$

where $(U_{\alpha/2}\,\Lambda^{2/\alpha}, \mu_1\Lambda)'$ has the Laplace transform (3.6). The proof is finished since the CMT gives:

$$\left(\frac{t}{a_t}\right)^2 T_{N(t)} = \left(\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2\right)\left(\frac{1}{t}\sum_{i=1}^{N(t)} X_i\right)^{-2} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\,\frac{U_{\alpha/2}}{\Lambda^{2-2/\alpha}} \quad \text{as} \quad t \to \infty.$$

Note that $(t/a_t)^2 \in \mathrm{RV}_{2-2/\alpha}$. $\square$

Theorem 3.4. *Assume that X_1 is of Pareto-type with index $\alpha = 2$ and $\mu_2 = \infty$. Assume that $\{N(t);\, t \geqslant 0\}$ $\mathcal{D}$-averages in time to the random variable Λ.*

(*a*) *Then:* $\left(\dfrac{N(t)}{a'_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \dfrac{2}{\mu_1^2}\,\Lambda$ *as* $t \to \infty$.

(*b*) *Then:* $\left(\dfrac{t}{a'_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \dfrac{2}{\mu_1^2}\,\dfrac{1}{\Lambda}$ *as* $t \to \infty$.

In (a) and (b), the sequence $(a'_t)_{t>0}$ is defined by $\lim_{t\to\infty} t\, a_t'^{-2}\tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,du \in \mathrm{RV}_0$.

Proof. Let $1 - F(x) \sim x^{-2}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$ such that $\mu_2 = \infty$. Define a sequence $(a'_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t'^{-2}\tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,du \in \mathrm{RV}_0$. Note that $a'_t \in \mathrm{RV}_{1/2}$ and then $(t/a'_t)^2 \in \mathrm{RV}_1$.

(*b*) Let $r \geqslant 0$ and $s \geqslant 0$ be fixed. We readily compute:

$$\mathbb{E}\left\{\exp\left(-r\frac{1}{a_t'^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{t}\sum_{i=1}^{N(t)} X_i\right)\right\} = Q_t\left(e^{-\delta_t(r,s)/t}\right)$$

with $\delta_t(r,s) := -t\,\log \int_0^\infty e^{-r\left(x/a'_t\right)^2 - sx/t}\,dF(x) \in [0,\infty)$. By virtue of the DCT, we have $\lim_{t\to\infty}\int_0^\infty e^{-r\left(x/a'_t\right)^2 - sx/t}\,dF(x) = 1$. It then follows that:

$$\begin{aligned}\delta_t(r,s) \underset{t\uparrow\infty}{\sim} & \frac{2r^2 t}{a_t'^2}\int_0^\infty e^{-ry}\int_0^{a'_t\sqrt{y}} x\,(1-F(x))\,e^{-sx/t}\,dx\,dy \\ & + s\int_0^\infty (1-F(x))\,e^{-r(x/a'_t)^2 - sx/t}\,dx.\end{aligned}$$

Since $\mu_2 = \infty$, we have $\tilde{\ell}(x) \sim \int_0^x u\,(1-F(u))\,du$ as $x \to \infty$. For any $y > 0$, we then obtain as $t \to \infty$:

$$\int_0^{a'_t\sqrt{y}} x\,(1-F(x))\,e^{-sx/t}\,dx \sim \int_0^{a'_t\sqrt{y}} x\,(1-F(x))\,dx \sim \tilde{\ell}\big(a'_t\sqrt{y}\big) \sim \tilde{\ell}(a'_t) \sim \frac{a_t'^2}{t}$$

so that the DCT leads to:

$$\lim_{t\to\infty}\frac{2r^2 t}{a_t'^2}\int_0^\infty e^{-ry}\int_0^{a'_t\sqrt{y}} x\,(1-F(x))\,e^{-sx/t}\,dx\,dy = 2r.$$

Since $\mu_1 < \infty$, we have by virtue of the DCT:

$$\lim_{t\to\infty} s\int_0^\infty (1-F(x))\,e^{-r\left(x/a'_t\right)^2 - sx/t}\,dx = s\mu_1.$$

It follows that $\lim_{t\to\infty}\delta_t(r,s) = 2r + s\mu_1$. A similar argument as in the proof of Theorem 3.1 applied to the convergence of $Q_t\bigl(e^{-\delta_t(r,s)/t}\bigr)$ then yields:

$$\lim_{t\to\infty}\mathbb{E}\biggl\{\exp\biggl(-r\frac{1}{a_t'^2}\sum_{i=1}^{N(t)}X_i^2 - s\frac{1}{t}\sum_{i=1}^{N(t)}X_i\biggr)\biggr\} = \mathbb{E}\bigl\{e^{-2r\Lambda - s\mu_1\Lambda}\bigr\}\,,\ r\geqslant 0,\ s\geqslant 0$$

or equivalently:

$$\left(\frac{1}{a_t'^2}\sum_{i=1}^{N(t)}X_i^2,\ \frac{1}{t}\sum_{i=1}^{N(t)}X_i\right)' \xrightarrow{\mathcal{D}} (2\Lambda, \mu_1\Lambda)' \quad \text{as} \quad t\to\infty. \tag{3.8}$$

The CMT finally gives:

$$\left(\frac{t}{a_t'}\right)^2 T_{N(t)} = \left(\frac{1}{a_t'^2}\sum_{i=1}^{N(t)}X_i^2\right)\left(\frac{1}{t}\sum_{i=1}^{N(t)}X_i\right)^{-2} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1^2}\frac{1}{\Lambda} \quad \text{as} \quad t\to\infty.$$

(a) Since $\mu_1<\infty$ and $N(t)\xrightarrow{a.s.}\infty$ as $t\to\infty$, we get $\frac{1}{N(t)}\sum_{i=1}^{N(t)}X_i \xrightarrow{p} \mu_1$ as $t\to\infty$ by Lemma 2.1. Moreover, it follows from (3.8) that $(a_t')^{-2}\sum_{i=1}^{N(t)}X_i^2 \xrightarrow{\mathcal{D}} 2\Lambda$ as $t\to\infty$. Slutsky's theorem and the CMT then lead to:

$$\left(\frac{N(t)}{a_t'}\right)^2 T_{N(t)} = \left(\frac{1}{a_t'^2}\sum_{i=1}^{N(t)}X_i^2\right)\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)}X_i\right)^{-2} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1^2}\Lambda \quad \text{as} \quad t\to\infty$$

and this ends the proof. □

When X_1 is of Pareto-type with index $\alpha>2$, we have $\mu_2<\infty$ so that $N(t)\,T_{N(t)} \xrightarrow{p} \mu_2/\mu_1^2$ as $t\to\infty$ by the law of large numbers and Lemma 2.1 since $N(t)\xrightarrow{a.s.}\infty$ as $t\to\infty$. In the sequel, we then derive second-order weak convergence results.

THEOREM 3.5. *Assume that X_1 is of Pareto-type with index $\alpha\in(2,4)$ (including $\alpha=2$ if $\mu_2<\infty$) and that $\{N(t);\, t\geqslant 0\}$ p-averages in time to the random variable Λ. Then:*

$$\frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}} \quad \textit{as} \quad t\to\infty$$

where $W_{\alpha/2}$ is an $\alpha/2$-stable random variable independent of Λ and where $\ell_1^{\#}\in\mathrm{RV}_0$ is the de Bruijn conjugate of $\ell_1(x) := \ell^{-2/\alpha}\bigl(\sqrt{x}\bigr)\in\mathrm{RV}_0$.

PROOF. Let $1-F(x)\sim x^{-\alpha}\ell(x)$ as $x\to\infty$ for some $\ell\in\mathrm{RV}_0$ and $\alpha\in(2,4)$ or $\alpha=2$ if $\mu_2<\infty$. Since $N(t)\xrightarrow{a.s.}\infty$ as $t\to\infty$, we combine Lemma 2.1 and Theorem 2.5 of Albrecher et al. [**1**] to obtain:

$$\frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})}\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}W_{\alpha/2} \quad \text{as} \quad t\to\infty \tag{3.9}$$

where $W_{\alpha/2}$ is a stable random variable with exponent $\alpha/2$ and $\ell_1^{\#}\in\mathrm{RV}_0$ is the de Bruijn conjugate of $\ell_1(x) := \ell^{-2/\alpha}(\sqrt{x})\in\mathrm{RV}_0$.

Let us prove the independence of $W_{\alpha/2}$ and Λ. We condition on $N(t)$, use the independence of $\{N(t);\, t \geqslant 0\}$ and $\{X_i;\, i \geqslant 1\}$ and finally apply (3.9) to get:

$$\begin{aligned} Y_t &:= \mathbb{P}\biggl[\frac{N(t)}{t} \leqslant x,\, \frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})}\bigl(\mu_1^2\, N(t)\, T_{N(t)} - \mu_2\bigr) \leqslant y \,\bigg|\, N(t)\biggr] \\ &= 1_{\{N(t)/t \leqslant x\}}\, \mathbb{P}\biggl[\frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})}\bigl(\mu_1^2\, N(t)\, T_{N(t)} - \mu_2\bigr) \leqslant y\biggr] \\ &\xrightarrow{\mathcal{D}} 1_{\{\Lambda \leqslant x\}}\, \mathbb{P}\bigl[W_{\alpha/2} \leqslant y\bigr] \quad \text{as} \quad t \to \infty \end{aligned}$$

at any continuity points x of the distribution function of Λ and y of that of $W_{\alpha/2}$. The sequence of random variables $\{Y_t;\, t > 0\}$ being uniformly integrable, we apply Theorem 5.4 of Billingsley [**3**] to obtain:

$$\lim_{t\to\infty} \mathbb{P}\biggl[\frac{N(t)}{t} \leqslant x,\, \frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})}\bigl(\mu_1^2\, N(t)\, T_{N(t)} - \mu_2\bigr) \leqslant y\biggr] = \mathbb{P}[\Lambda \leqslant x]\, \mathbb{P}[W_{\alpha/2} \leqslant y].$$

Now, since $\ell_1^{\#} \in \mathrm{RV}_0$ and $\frac{N(t)}{t} \xrightarrow{p} \Lambda$ as $t \to \infty$ with $\mathbb{P}[\Lambda > 0] = 1$, we have $\frac{\ell_1^{\#}(N(t)^{2/\alpha})}{\ell_1^{\#}(t^{2/\alpha})} \xrightarrow{p} 1$ as $t \to \infty$ by the uniform convergence theorem for slowly varying functions (e.g., Theorem 1.2.1 of Bingham et al. [**4**]), the CMT and the subsequence principle. Recalling (3.9), Slutsky's theorem and the CMT therefore yield as $t \to \infty$:

$$\begin{aligned} &\frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}\left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \\ &= \frac{\ell_1^{\#}(N(t)^{2/\alpha})}{\ell_1^{\#}(t^{2/\alpha})}\left(\frac{t}{N(t)}\right)^{1-2/\alpha} \frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})}\left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\, \frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}} \end{aligned}$$

thanks to the independence of $W_{\alpha/2}$ and Λ. The proof is complete. □

THEOREM 3.6. *Assume that X_1 is of Pareto-type with index $\alpha = 4$ and $\mu_4 = \infty$. Assume that $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Then:*

$$\frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})}\left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\, \frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}} \quad \textit{as} \quad t \to \infty$$

where $\mathrm{N}(0,1)$ is a standard normal random variable independent of Λ and where $\ell_2^{\#} \in \mathrm{RV}_0$ is the de Bruijn conjugate of $\ell_2(x) := \frac{1}{2\sqrt{\tilde{\ell}(\sqrt{x})}} \in \mathrm{RV}_0$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, du \in \mathrm{RV}_0$.

PROOF. Let $1 - F(x) \sim x^{-4}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$ such that $\mu_4 = \infty$. The proof is akin to that of Theorem 3.5. Since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, we combine Lemma 2.1 and Theorem 2.6 of Albrecher et al. [**1**] to obtain:

$$\frac{\sqrt{N(t)}}{\ell_2^{\#}\bigl(\sqrt{N(t)}\bigr)}\left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\, \mathrm{N}(0,1) \quad \text{as} \quad t \to \infty \tag{3.10}$$

where $\mathrm{N}(0,1)$ is a standard normal random variable and $\ell_2^{\#} \in \mathrm{RV}_0$ is the de Bruijn conjugate of $\ell_2(x) := \frac{1}{2\sqrt{\tilde{\ell}(\sqrt{x})}} \in \mathrm{RV}_0$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, du \in \mathrm{RV}_0$.

The random variables $\mathrm{N}(0,1)$ and Λ are independent. This is proved as for the independence of $W_{\alpha/2}$ and Λ in the proof of Theorem 3.5. Since $\ell_2^{\#} \in \mathrm{RV}_0$ and $\frac{N(t)}{t} \xrightarrow{p} \Lambda$ as $t \to \infty$ with $\mathbb{P}[\Lambda > 0] = 1$, we also have $\frac{\ell_2^{\#}(\sqrt{N(t)})}{\ell_2^{\#}(\sqrt{t})} \xrightarrow{p} 1$ as $t \to \infty$. Recalling (3.10), Slutsky's theorem and the CMT therefore yield as $t \to \infty$:

$$\begin{aligned} &\frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})}\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \\ &\qquad = \frac{\ell_2^{\#}\big(\sqrt{N(t)}\big)}{\ell_2^{\#}\big(\sqrt{t}\big)} \sqrt{\frac{t}{N(t)}}\, \frac{\sqrt{N(t)}}{\ell_2^{\#}\big(\sqrt{N(t)}\big)} \left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\,\frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}} \end{aligned}$$

and the proof is finished. $\square$

The following theorem covers the remaining α-cases since the result applies in particular when X_1 is of Pareto-type with index $\alpha = 4$ if $\mu_4 < \infty$ or $\alpha > 4$.

THEOREM 3.7. *Assume that* X_1 *satisfies* $\mu_4 < \infty$ *and that* $\{N(t);\, t \geqslant 0\}$ $\mathcal{D}$-*averages in time to the random variable* Λ. *Then:*

$$\sqrt{t}\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{\mathrm{N}(0,\sigma_*^2)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty$$

where $\mathrm{N}(0,\sigma_*^2)$ *is a normal random variable independent of* Λ *with mean* 0 *and variance* σ_*^2 *defined by:*

$$\sigma_*^2 := \frac{\mu_4}{\mu_1^4} - \left(\frac{\mu_2}{\mu_1^2}\right)^2 + 4\left(\frac{\mu_2}{\mu_1^2}\right)^3 - \frac{4\mu_2\mu_3}{\mu_1^5}. \tag{3.11}$$

PROOF. Let the distribution function F of X_1 be such that $\mu_4 < \infty$. From the bivariate Lindeberg-Lévy central limit theorem (e.g., Theorem 1.9.1B of Serfling [**9**]), one deduces that:

$$\sqrt{n}\bigg(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y_n} - \boldsymbol{\mu}\bigg) \xrightarrow{\mathcal{D}} \mathrm{N}(\mathbf{0},\boldsymbol{\Sigma}) \quad \text{as} \quad n \to \infty \tag{3.12}$$

where $\boldsymbol{Y_n} := \big(X_i, X_i^2\big)'$, $\boldsymbol{\mu} := (\mu_1,\mu_2)'$ and $\mathrm{N}(\mathbf{0},\boldsymbol{\Sigma})$ is a normal random vector with mean $\mathbf{0} := (0,0)'$ and covariance matrix $\boldsymbol{\Sigma}$ defined by:

$$\boldsymbol{\Sigma} := \begin{pmatrix} \mu_2 - \mu_1^2 & \mu_3 - \mu_1\mu_2 \\ \mu_3 - \mu_1\mu_2 & \mu_4 - \mu_2^2 \end{pmatrix}.$$

Following the notation in Serfling [**9**], we write (3.12) as $\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y_n}$ is $\mathrm{AN}\big(\boldsymbol{\mu}, n^{-1}\boldsymbol{\Sigma}\big)$. By virtue of the multivariate delta method, the asymptotic normality carries over to the random variable $g\big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y_n}\big) = g\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n} X_i^2\right)$ for any function $g\colon (0,\infty)\times(0,\infty) \to \mathbb{R}$ that is continuously differentiable in a neighborhood of $\boldsymbol{\mu}$, so that $g\big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y_n}\big)$ is $\mathrm{AN}\big(g(\boldsymbol{\mu}), n^{-1}\boldsymbol{J\Sigma J'}\big)$ with $\boldsymbol{J} := \Big(\frac{\partial g}{\partial x}(\boldsymbol{\mu}), \frac{\partial g}{\partial y}(\boldsymbol{\mu})\Big)$. With the choice $g(x,y) = y/x^2$, we find that nT_n is $\mathrm{AN}\Big(\frac{\mu_2}{\mu_1^2}, \frac{\sigma_*^2}{n}\Big)$ with σ_*^2 given by (3.11).

Since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, it consequently follows by Lemma 2.1 that:

$$\sqrt{N(t)}\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \mathrm{N}(0, \sigma_*^2) \quad \text{as} \quad t \to \infty$$

where $\mathrm{N}(0, \sigma_*^2)$ is a normal random variable with mean 0 and variance σ_*^2. The CMT together with the independence of $\mathrm{N}(0, \sigma_*^2)$ and Λ (which is proved using the same arguments as for the independence of $W_{\alpha/2}$ and Λ in the proof of Theorem 3.5) finally gives as $t \to \infty$:

$$\sqrt{t}\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) = \sqrt{\frac{t}{N(t)}}\,\sqrt{N(t)}\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{\mathrm{N}(0, \sigma_*^2)}{\sqrt{\Lambda}}.$$

This completes the proof. □

4. Applications to Risk Measures

Assume that X is a positive random variable with distribution function F and let $X_1, \ldots, X_{N(t)}$ be a random sample from F of random size $N(t)$ from a nonnegative integer-valued distribution. Thanks to the limiting results derived in Section 3 and the relations (1.4) and (1.7), we investigate the asymptotic behavior of two popular risk measures through their distributions. Subsection 4.1 deals with the sample coefficient of variation $\widehat{\mathrm{CoVar}(X)}$ defined in (1.3) and Subsection 4.2 concerns the sample dispersion $\widehat{\mathrm{D}(X)}$ defined in (1.5). The results are obtained under the same assumptions on X and on the counting process $\{N(t);\, t \geqslant 0\}$ as in Section 3.

4.1. Sample Coefficient of Variation. We determine limits in distribution for the appropriately normalized random variable $\widehat{\mathrm{CoVar}(X)}$ by using the distributional results derived in Section 3 for $T_{N(t)}$ and thanks to (1.4). Consequently, different cases will arise according to the range of α and the (non)finiteness of the first few moments. We assume that X is of Pareto-type with index $\alpha > 0$ as defined in (1.2) in Cases 1-6 and that X satisfies $\mu_4 < \infty$ in Case 7. Moreover, the counting process is supposed to $\mathcal{D}$-average in time to the random variable Λ except in Cases 5-6 where it p-averages in time to Λ.

<u>Case 1:</u> $\alpha \in (0, 1)$. Since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, it follows from Theorem 3.1, Slutsky's theorem and the CMT that as $t \to \infty$:

$$\frac{\widehat{\mathrm{CoVar}(X)}}{\sqrt{N(t)}} = \sqrt{T_{N(t)} - \frac{1}{N(t)}} \xrightarrow{\mathcal{D}} \frac{\sqrt{U_{\alpha/2}}}{U_\alpha}$$

where the distribution of the random vector $(U_{\alpha/2}, U_\alpha)'$ is determined by (3.1).

<u>Case 2:</u> $\alpha = 1$, $\mu_1 = \infty$. Define $(a_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t^{-1} \ell(a_t) = 1$ and $(a'_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t'^{-1} \tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, du \in \mathrm{RV}_0$. Since $\frac{a'_t}{a_t} \sim \frac{(1/\tilde{\ell})^{\#}(t)}{(1/\ell)^{\#}(t)}$ as $t \to \infty$, where $(1/\tilde{\ell})^{\#} \in \mathrm{RV}_0$ and $(1/\ell)^{\#} \in \mathrm{RV}_0$ are the de Bruijn conjugates of $1/\tilde{\ell} \in \mathrm{RV}_0$ and $1/\ell \in \mathrm{RV}_0$ respectively, it follows that $a'_t/a_t \in \mathrm{RV}_0$ and then that

$\lim_{t\to\infty} \frac{1}{t}\big(\frac{a'_t}{a_t}\big)^2 = 0$. Moreover, $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$. Hence, Theorem 3.2 together with Slutsky's theorem and the CMT gives as $t \to \infty$:

$$\frac{a'_t}{a_t}\,\frac{\widehat{\mathrm{CoVar}}(X)}{\sqrt{N(t)}} = \sqrt{\left(\frac{a'_t}{a_t}\right)^2 T_{N(t)} - \frac{1}{t}\left(\frac{a'_t}{a_t}\right)^2 \frac{t}{N(t)}} \xrightarrow{\mathcal{D}} \sqrt{U_{1/2}}$$

where the distribution of the random variable $U_{1/2}$ is determined by (2.2) with $\gamma = 1/2$.

Case 3: $\alpha \in (1,2)$ or $\alpha = 1$, $\mu_1 < \infty$. Define $(a_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t^{-\alpha}\ell(a_t) = 1$. Since $\frac{t}{a_t^2} \sim \frac{a_t^{\alpha-2}}{\ell(a_t)} \to 0$ and $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$, it follows from Theorem 3.3(a), Slutsky's theorem and the CMT that as $t \to \infty$:

$$\frac{\sqrt{N(t)}}{a_t}\,\widehat{\mathrm{CoVar}}(X) = \sqrt{\left(\frac{N(t)}{a_t}\right)^2 T_{N(t)} - \frac{t}{a_t^2}\,\frac{N(t)}{t}} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\sqrt{U_{\alpha/2}}\,\Lambda^{1/\alpha}$$

where the random variable $U_{\alpha/2}$ is independent of Λ and has a distribution determined by (2.2) with $\gamma = \alpha/2$.

Repeating the same arguments as above but using Theorem 3.3(b) instead of Theorem 3.3(a), we also get as $t \to \infty$:

$$\frac{t}{a_t}\,\frac{\widehat{\mathrm{CoVar}}(X)}{\sqrt{N(t)}} = \sqrt{\left(\frac{t}{a_t}\right)^2 T_{N(t)} - \frac{t}{a_t^2}\,\frac{t}{N(t)}} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\,\frac{\sqrt{U_{\alpha/2}}}{\Lambda^{1-1/\alpha}}.$$

Case 4: $\alpha = 2$, $\mu_2 = \infty$. Define $(a'_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t'^{-2}\tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,du \in \mathrm{RV}_0$. From $\mu_2 = \infty$, it follows that $\lim_{x\to\infty} \tilde{\ell}(x) = \infty$ so that $t/a_t'^2 \sim 1/\tilde{\ell}(a'_t) \to 0$ as $t \to \infty$. Moreover, $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$. Theorem 3.4(a), Slutsky's theorem and the CMT then yield as $t \to \infty$:

$$\frac{\sqrt{N(t)}}{a'_t}\,\widehat{\mathrm{CoVar}}(X) = \sqrt{\left(\frac{N(t)}{a'_t}\right)^2 T_{N(t)} - \frac{t}{a_t'^2}\,\frac{N(t)}{t}} \xrightarrow{\mathcal{D}} \frac{\sqrt{2}}{\mu_1}\sqrt{\Lambda}.$$

By using Theorem 3.4(b) and the arguments above, we also get as $t \to \infty$:

$$\frac{t}{a'_t}\,\frac{\widehat{\mathrm{CoVar}}(X)}{\sqrt{N(t)}} = \sqrt{\left(\frac{t}{a'_t}\right)^2 T_{N(t)} - \frac{t}{a_t'^2}\,\frac{t}{N(t)}} \xrightarrow{\mathcal{D}} \frac{\sqrt{2}}{\mu_1}\,\frac{1}{\sqrt{\Lambda}}.$$

Case 5: $\alpha \in (2,4)$ or $\alpha = 2$, $\mu_2 < \infty$. Assume that $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Let $\ell_1^{\#} \in \mathrm{RV}_0$ be the de Bruijn conjugate of $\ell_1(x) := \ell^{-2/\alpha}(\sqrt{x}) \in \mathrm{RV}_0$. Note that $\ell_1^{\#}(x) = o(1)$ as $x \to \infty$ if $\alpha = 2$ and $\mu_2 < \infty$ since $\ell(x) = o(1)$ as $x \to \infty$. Since $N(t)\,T_{N(t)} \xrightarrow{p} \mu_2/\mu_1^2$ as $t \to \infty$, the CMT gives:

$$\widehat{\mathrm{CoVar}}(X) \xrightarrow{p} \mathrm{CoVar}(X) \quad \text{as} \quad t \to \infty.$$

Now, define a sequence $(b_t)_{t>0}$ by $b_t := \frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}$. Let $\sigma^2 := \mathbb{V}X < \infty$ and consider:

$$b_t\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big) = \underbrace{\frac{\mu_1}{2\sigma}\, b_t\Big(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2}\Big)}_{=:A_t} - \underbrace{\frac{\mu_1 b_t\Big(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2}\Big)^2}{2\sigma\Big(\widehat{\mathrm{CoVar}(X)} + \mathrm{CoVar}(X)\Big)^2}}_{=:B_t}.$$

From Theorem 3.5 and using Slutsky's theorem and the CMT, we easily deduce that $A_t \xrightarrow{\mathcal{D}} \frac{1}{2\mu_1\sigma}\frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}}$ and $B_t \xrightarrow{p} 0$ as $t \to \infty$, leading by virtue of another application of Slutsky's theorem to:

$$\frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big) \xrightarrow{\mathcal{D}} \frac{1}{2\mu_1\sigma}\frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}} \quad \text{as} \quad t \to \infty$$

where $W_{\alpha/2}$ is an $\alpha/2$-stable random variable independent of Λ.

<u>Case 6:</u> $\alpha = 4$, $\mu_4 = \infty$. Assume that $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Since $N(t)\, T_{N(t)} \xrightarrow{p} \mu_2/\mu_1^2$ as $t \to \infty$, we deduce by an application of the CMT that:

$$\widehat{\mathrm{CoVar}(X)} \xrightarrow{p} \mathrm{CoVar}(X) \quad \text{as} \quad t \to \infty.$$

Now, let $\ell_2(x) := \frac{1}{2\sqrt{\tilde{\ell}(\sqrt{x})}} \in \mathrm{RV}_0$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, du \in \mathrm{RV}_0$. Define a sequence $(c_t)_{t>0}$ by $c_t := \frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})}$ where $\ell_2^{\#} \in \mathrm{RV}_0$ is the de Bruijn conjugate of ℓ_2. Consider the following equality:

$$c_t\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big) =: A_t - B_t$$

where the random variables A_t and B_t are defined as in Case 5 but with b_t replaced by c_t. Theorem 3.6, Slutsky's theorem and the CMT give $A_t \xrightarrow{\mathcal{D}} \frac{1}{2\mu_1\sigma}\frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}}$ and $B_t \xrightarrow{p} 0$ as $t \to \infty$, leading by another application of Slutsky's theorem to:

$$\frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})}\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big) \xrightarrow{\mathcal{D}} \frac{1}{2\mu_1\sigma}\frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty$$

where $\mathrm{N}(0,1)$ is a standard normal random variable independent of Λ.

<u>Case 7:</u> $\mu_4 < \infty$. The proof of Theorem 3.7 can be repeated using the transformation $g(x,y) = \sqrt{y/x^2 - 1}$ and this leads to:

$$\sqrt{t}\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big) \xrightarrow{\mathcal{D}} \frac{\mathrm{N}\Big(0, \frac{\sigma_*^2\mu_1^2}{4\sigma^2}\Big)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty \tag{4.1}$$

where $\mathrm{N}\big(0, \sigma_*^2\mu_1^2/(4\sigma^2)\big)$ is a normal random variable independent of Λ with mean 0 and variance $\sigma_*^2\mu_1^2/(4\sigma^2)$, with σ_*^2 defined in (3.11) and $\sigma^2 := \mathbb{V}X < \infty$.

Assume that $\mathbb{E}\{\Lambda^{-1}\} < \infty$. When $t\big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\big)^2$ is uniformly integrable, the first two moments of the limiting distribution in (4.1) permit to determine the limiting behavior of $\mathrm{CoVar}\big(\widehat{\mathrm{CoVar}(X)}\big)$. Indeed, on the one hand:

$$\lim_{t\to\infty}\sqrt{t}\Big(\mathbb{E}\Big\{\widehat{\mathrm{CoVar}(X)}\Big\}-\mathrm{CoVar}(X)\Big)=\lim_{t\to\infty}\mathbb{E}\Big\{\sqrt{t}\Big(\widehat{\mathrm{CoVar}(X)}-\mathrm{CoVar}(X)\Big)\Big\}=0$$

which leads to:

$$\lim_{t\to\infty}\mathbb{E}\Big\{\widehat{\mathrm{CoVar}(X)}\Big\}=\mathrm{CoVar}(X). \tag{4.2}$$

On the other hand, we also get:

$$\lim_{t\to\infty}t\,\mathbb{V}\Big\{\widehat{\mathrm{CoVar}(X)}\Big\}=\lim_{t\to\infty}\mathbb{V}\Big\{\sqrt{t}\Big(\widehat{\mathrm{CoVar}(X)}-\mathrm{CoVar}(X)\Big)\Big\}=\frac{\sigma_*^2\mu_1^2\,\mathbb{E}\{\Lambda^{-1}\}}{4\sigma^2}$$

so that:

$$\mathbb{V}\Big\{\widehat{\mathrm{CoVar}(X)}\Big\}\sim\frac{\sigma_*^2\mu_1^2\,\mathbb{E}\{\Lambda^{-1}\}}{4\sigma^2}\,\frac{1}{t}\quad\text{as}\quad t\to\infty.$$

Consequently, under the above uniform integrability condition, the coefficient of variation of the sample coefficient of variation asymptotically behaves as:

$$\mathrm{CoVar}\Big(\widehat{\mathrm{CoVar}(X)}\Big)\sim\frac{\sigma_*\mu_1^2\,\sqrt{\mathbb{E}\{\Lambda^{-1}\}}}{2\sigma^2}\,\frac{1}{\sqrt{t}}\quad\text{as}\quad t\to\infty.$$

In addition, it results from (4.1) and (4.2) that $\widehat{\mathrm{CoVar}(X)}$ is a consistent and asymptotically unbiased estimator for $\mathrm{CoVar}(X)$.

4.2. Sample Dispersion. Adapting the results of Section 3 to the random variable $C_{N(t)}$ defined in (1.6) permits us to derive limiting distributions for the appropriately normalized sample dispersion $\widehat{\mathrm{D}(X)}$ from (1.7). Different cases are considered as for the sample coefficient of variation. We assume that X is of Pareto-type with index $\alpha>0$ as defined in (1.2) in Cases 1-6 and that X satisfies $\mu_4<\infty$ in Case 7. Moreover, the counting process is supposed to $\mathcal{D}$-average in time to the random variable Λ except in Cases 5-6 where it p-averages in time to Λ.

Case 1: $\alpha\in(0,1)$. Define $(a_t)_{t>0}$ by $\lim_{t\to\infty}t\,a_t^{-\alpha}\ell(a_t)=1$. It follows from the CMT and (3.3) that as $t\to\infty$:

$$\frac{1}{a_t}\,C_{N(t)}\xrightarrow{\mathcal{D}}\frac{U_{\alpha/2}}{U_\alpha}\,\Lambda^{1/\alpha}$$

where the random vector $(U_{\alpha/2},U_\alpha)'$ is independent of Λ and has a distribution determined by (3.1). Since $N(t)\xrightarrow{a.s.}\infty$ and $a_t^{-1}\sum_{i=1}^{N(t)}X_i\xrightarrow{\mathcal{D}}U_\alpha\,\Lambda^{1/\alpha}$ as $t\to\infty$, Slutsky's theorem and the CMT then yield as $t\to\infty$:

$$\frac{1}{a_t}\,\widehat{\mathrm{D}(X)}=\frac{1}{a_t}\,C_{N(t)}-\frac{1}{N(t)}\frac{1}{a_t}\sum_{i=1}^{N(t)}X_i\xrightarrow{\mathcal{D}}\frac{U_{\alpha/2}}{U_\alpha}\,\Lambda^{1/\alpha}.$$

Case 2: $\alpha = 1$, $\mu_1 = \infty$. Define $(a_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t^{-1}\ell(a_t) = 1$ and $(a'_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t'^{-1}\tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, du \in \mathrm{RV}_0$. It follows from (3.5) and the CMT that as $t \to \infty$:

$$\frac{a'_t}{a_t^2}\, C_{N(t)} \xrightarrow{\mathcal{D}} U_{1/2}\, \Lambda$$

where the random variable $U_{1/2}$ is independent of Λ and has a distribution determined by (2.2) with $\gamma = 1/2$.

Since $\frac{a'_t}{a_t} \sim \frac{(1/\tilde{\ell})^{\#}(t)}{(1/\ell)^{\#}(t)}$ as $t \to \infty$, where $(1/\tilde{\ell})^{\#} \in \mathrm{RV}_0$ and $(1/\ell)^{\#} \in \mathrm{RV}_0$ are the de Bruijn conjugates of $1/\tilde{\ell} \in \mathrm{RV}_0$ and $1/\ell \in \mathrm{RV}_0$ respectively, it follows that $a'_t/a_t \in \mathrm{RV}_0$ and then that $\lim_{t\to\infty} t^{-1}(a'_t/a_t)^2 = 0$. Moreover, using the same independence and conditioning arguments as in the proof of Theorem 3.5, we obtain that at any continuity points x and y of the distribution function of Λ:

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{N(t)}{t} \leqslant x, \frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i \leqslant y\right] = \mathbb{P}[\Lambda \leqslant x]\, \mathbb{P}[\Lambda \leqslant y]$$

i.e., since $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ and $(a'_t)^{-1}\sum_{i=1}^{N(t)} X_i \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$, that:

$$\left(\frac{N(t)}{t}, \frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i\right)' \xrightarrow{\mathcal{D}} (\Lambda, \Lambda^\star)' \quad \text{as} \quad t \to \infty$$

where $\Lambda^\star$ is an independent copy of Λ. Using the CMT, we then deduce:

$$\frac{t}{N(t)} \frac{\sum_{i=1}^{N(t)} X_i}{a'_t} \xrightarrow{\mathcal{D}} \frac{\Lambda^\star}{\Lambda} \quad \text{as} \quad t \to \infty.$$

Hence, Slutsky's theorem gives as $t \to \infty$:

$$\frac{a'_t}{a_t^2}\, \widehat{\mathrm{D}(X)} = \frac{a'_t}{a_t^2}\, C_{N(t)} - \frac{1}{t}\left(\frac{a'_t}{a_t}\right)^2 \frac{t}{N(t)} \frac{\sum_{i=1}^{N(t)} X_i}{a'_t} \xrightarrow{\mathcal{D}} U_{1/2}\, \Lambda.$$

Case 3: $\alpha \in (1,2)$ or $\alpha = 1$, $\mu_1 < \infty$. Define $(a_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t^{-\alpha}\ell(a_t) = 1$. Since $\overline{X} \xrightarrow{p} \mu_1$ as $t \to \infty$, we get from Theorem 3.3(a) and Slutsky's theorem that:

$$\frac{N(t)}{a_t^2}\, C_{N(t)} = \overline{X}\left(\frac{N(t)}{a_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\, U_{\alpha/2}\, \Lambda^{2/\alpha} \quad \text{as} \quad t \to \infty$$

where the random variable $U_{\alpha/2}$ is independent of Λ and has a distribution determined by (2.2) with $\gamma = \alpha/2$. Since $\frac{t}{a_t^2} \sim \frac{a_t^{\alpha-2}}{\ell(a_t)} \to 0$ and $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$, Slutsky's theorem gives as $t \to \infty$:

$$\frac{N(t)}{a_t^2}\, \widehat{\mathrm{D}(X)} = \frac{N(t)}{a_t^2}\, C_{N(t)} - \frac{t}{a_t^2}\frac{N(t)}{t}\, \overline{X} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\, U_{\alpha/2}\, \Lambda^{2/\alpha}.$$

By using (3.7) and the arguments above, we also get as $t \to \infty$:

$$\frac{t}{a_t^2}\, \widehat{\mathrm{D}(X)} = \frac{t}{a_t^2}\, C_{N(t)} - \frac{t}{a_t^2}\, \overline{X} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1} \frac{U_{\alpha/2}}{\Lambda^{1-2/\alpha}}.$$

Case 4: $\alpha = 2$, $\mu_2 = \infty$. Define $(a'_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t'^{-2}\tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,du \in \mathrm{RV}_0$. Since $\overline{X} \xrightarrow{p} \mu_1$ as $t \to \infty$, it follows from Theorem 3.4(a) and Slutsky's theorem that as $t \to \infty$:

$$\frac{N(t)}{a_t'^2}\, C_{N(t)} = \overline{X}\left(\frac{N(t)}{a'_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1}\,\Lambda.$$

From $\mu_2 = \infty$, we get $\lim_{x\to\infty}\tilde{\ell}(x) = \infty$ so that $\frac{t}{a_t'^2} \sim \frac{1}{\tilde{\ell}(a'_t)} \to 0$ as $t \to \infty$. Since $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$, Slutsky's theorem then yields as $t \to \infty$:

$$\frac{N(t)}{a_t'^2}\,\widehat{\mathrm{D}(X)} = \frac{N(t)}{a_t'^2}\, C_{N(t)} - \frac{t}{a_t'^2}\,\frac{N(t)}{t}\,\overline{X} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1}\,\Lambda.$$

By using (3.8) and the arguments above, we also get as $t \to \infty$:

$$\frac{t}{a_t'^2}\,\widehat{\mathrm{D}(X)} = \frac{t}{a_t'^2}\, C_{N(t)} - \frac{t}{a_t'^2}\,\overline{X} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1}.$$

Case 5: $\alpha \in (2, 4)$ or $\alpha = 2$, $\mu_2 < \infty$. Assume that $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Define a sequence $(b_t)_{t>0}$ by $b_t := \frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}$ where $\ell_1^{\#} \in \mathrm{RV}_0$ is the de Bruijn conjugate of $\ell_1(x) := \ell^{-2/\alpha}(\sqrt{x}) \in \mathrm{RV}_0$. Note that if $\alpha = 2$ and $\mu_2 < \infty$, we have $\ell_1^{\#}(x) = o(1)$ as $x \to \infty$. Consider the decomposition:

$$b_t\Big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\Big) = \underbrace{\frac{b_t}{\overline{X}}\bigg(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i^2 - \mu_2\bigg)}_{=:A_t} - \underbrace{\frac{b_t}{\sqrt{t}}\,\sqrt{t}\bigg(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i - \mu_1\bigg)\bigg(1 + \frac{\mu_2}{\mu_1}\frac{1}{\overline{X}}\bigg)}_{=:B_t}.$$

By using (3.9), it is readily proved that:

$$\frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})}\bigg(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i^2 - \mu_2\bigg) \xrightarrow{\mathcal{D}} W_{\alpha/2} \quad \text{as} \quad t \to \infty$$

where $W_{\alpha/2}$ is an $\alpha/2$-stable random variable independent of Λ. Since $\overline{X} \xrightarrow{p} \mu_1$, $\frac{N(t)}{t} \xrightarrow{p} \Lambda$ and $\frac{\ell_1^{\#}(N(t)^{2/\alpha})}{\ell_1^{\#}(t^{2/\alpha})} \xrightarrow{p} 1$ as $t \to \infty$, Slutsky's theorem and the CMT therefore give as $t \to \infty$:

$$A_t = \frac{1}{\overline{X}}\left(\frac{t}{N(t)}\right)^{1-2/\alpha} \frac{\ell_1^{\#}(N(t)^{2/\alpha})}{\ell_1^{\#}(t^{2/\alpha})}\,\frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})}\bigg(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i^2 - \mu_2\bigg) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\,\frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}}.$$

Using that $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, we combine the central limit theorem and Lemma 2.1 to obtain:

$$\sqrt{N(t)}\bigg(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i - \mu_1\bigg) \xrightarrow{\mathcal{D}} \mathrm{N}(0, \sigma^2) \quad \text{as} \quad t \to \infty$$

where the random variable $\mathrm{N}(0, \sigma^2)$ is normally distributed with mean 0 and variance $\sigma^2 := \mathbb{V}X < \infty$. The CMT together with the independence of $\mathrm{N}(0, \sigma^2)$ and Λ

(which is easily proved using the same kind of arguments as for the independence of $W_{\alpha/2}$ and Λ in the proof of Theorem 3.5) then yields as $t \to \infty$:

$$\sqrt{t}\Big(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i - \mu_1\Big) = \sqrt{\frac{t}{N(t)}}\sqrt{N(t)}\Big(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i - \mu_1\Big) \xrightarrow{\mathcal{D}} \frac{\mathrm{N}(0,\sigma^2)}{\sqrt{\Lambda}}. \tag{4.3}$$

Since $\lim_{t\to\infty} b_t/\sqrt{t} = 0$ and $\overline{X} \xrightarrow{p} \mu_1$ as $t \to \infty$, Slutsky's theorem and the CMT then imply that $B_t \xrightarrow{p} 0$ as $t \to \infty$. By virtue of another application of Slutsky's theorem, we finally obtain:

$$\frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}\left(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\,\frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}} \quad \text{as} \quad t \to \infty.$$

The latter relation shows in particular that:

$$\widehat{\mathrm{D}(X)} \xrightarrow{p} \mathrm{D}(X) \quad \text{as} \quad t \to \infty.$$

<u>Case 6:</u> $\alpha = 4$, $\mu_4 = \infty$. Assume that $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Define a sequence $(c_t)_{t>0}$ by $c_t := \frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})}$ where $\ell_2^{\#} \in \mathrm{RV}_0$ is the de Bruijn conjugate of $\ell_2(x) := \frac{1}{2\sqrt{\tilde{\ell}(\sqrt{x})}} \in \mathrm{RV}_0$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,du \in \mathrm{RV}_0$. Consider the following equality:

$$c_t\left(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\right) =: A_t - B_t$$

where the random variables A_t and B_t are defined as in Case 5 but with b_t replaced by c_t. By using (3.10), we get:

$$\frac{\sqrt{N(t)}}{\ell_2^{\#}(\sqrt{N(t)})}\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i^2 - \mu_2\right) \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \quad \text{as} \quad t \to \infty$$

for a standard normal random variable $\mathrm{N}(0,1)$ independent of Λ. Since $\overline{X} \xrightarrow{p} \mu_1$, $\frac{N(t)}{t} \xrightarrow{p} \Lambda$ and $\frac{\ell_2^{\#}(\sqrt{N(t)})}{\ell_2^{\#}(\sqrt{t})} \xrightarrow{p} 1$ as $t \to \infty$, Slutsky's theorem and the CMT then give as $t \to \infty$:

$$A_t = \frac{1}{\overline{X}}\sqrt{\frac{t}{N(t)}}\,\frac{\ell_2^{\#}(\sqrt{N(t)})}{\ell_2^{\#}(\sqrt{t})}\,\frac{\sqrt{N(t)}}{\ell_2^{\#}(\sqrt{N(t)})}\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i^2 - \mu_2\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\,\frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}}.$$

Since $\lim_{x\to\infty} \tilde{\ell}(x) = \infty$, we have $\lim_{x\to\infty} \ell_2(x) = 0$ so that $\lim_{x\to\infty} \ell_2^{\#}(x) = \infty$. Since $\overline{X} \xrightarrow{p} \mu_1$ as $t \to \infty$ and by using (4.3), we therefore have by virtue of Slutsky's theorem and the CMT that:

$$B_t = \frac{1}{\ell_2^{\#}(\sqrt{t})}\,\sqrt{t}\Big(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i - \mu_1\Big)\left(1 + \frac{\mu_2}{\mu_1}\frac{1}{\overline{X}}\right) \xrightarrow{p} 0 \quad \text{as} \quad t \to \infty.$$

Another application of Slutsky's theorem finally gives:

$$\frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})}\left(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\,\frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty.$$

It follows in particular from the latter relation that:

$$\widehat{\mathrm{D}(X)} \xrightarrow{p} \mathrm{D}(X) \quad \text{as} \quad t \to \infty.$$

Case 7: $\mu_4 < \infty$. Using $g(x,y) = \frac{y}{x} - x$ in the proof of Theorem 3.7 yields:

$$\sqrt{t}\left(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\right) \xrightarrow{\mathcal{D}} \frac{\mathrm{N}\big(0, \sigma_{**}^2\big)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty \tag{4.4}$$

where $\mathrm{N}\big(0,\sigma_{**}^2\big)$ is a normal random variable independent of Λ with mean 0 and variance σ_{**}^2 defined by:

$$\sigma_{**}^2 := \mu_2 - \mu_1^2 + \frac{\mu_2^3}{\mu_1^4} - 2\frac{\mu_3}{\mu_1} - 2\frac{\mu_2\mu_3}{\mu_1^3} + 2\left(\frac{\mu_2}{\mu_1}\right)^2 + \frac{\mu_4}{\mu_1^2}.$$

Assume that $\mathbb{E}\{\Lambda^{-1}\} < \infty$. When $t\big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\big)^2$ is uniformly integrable, the first two moments of the limiting distribution in (4.4) permit to determine the limiting behavior of $\mathrm{D}\big(\widehat{\mathrm{D}(X)}\big)$. Indeed, on the one hand:

$$\lim_{t\to\infty} \sqrt{t}\Big(\mathbb{E}\Big\{\widehat{\mathrm{D}(X)}\Big\} - \mathrm{D}(X)\Big) = \lim_{t\to\infty} \mathbb{E}\Big\{\sqrt{t}\Big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\Big)\Big\} = 0$$

leading to:

$$\lim_{t\to\infty} \mathbb{E}\Big\{\widehat{\mathrm{D}(X)}\Big\} = \mathrm{D}(X). \tag{4.5}$$

Note that (4.4) together with (4.5) implies that $\widehat{\mathrm{D}(X)}$ is a consistent and asymptotically unbiased estimator for $\mathrm{D}(X)$. On the other hand, we also get:

$$\lim_{t\to\infty} t\,\mathbb{V}\Big\{\widehat{\mathrm{D}(X)}\Big\} = \lim_{t\to\infty} \mathbb{V}\Big\{\sqrt{t}\Big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\Big)\Big\} = \sigma_{**}^2\, \mathbb{E}\big\{\Lambda^{-1}\big\}$$

so that:

$$\mathbb{V}\Big\{\widehat{\mathrm{D}(X)}\Big\} \sim \sigma_{**}^2\, \mathbb{E}\big\{\Lambda^{-1}\big\}\,\frac{1}{t} \quad \text{as} \quad t \to \infty.$$

Consequently, under the above uniform integrability condition, the dispersion of the sample dispersion asymptotically behaves as:

$$\mathrm{D}\Big(\widehat{\mathrm{D}(X)}\Big) \sim \frac{\sigma_{**}^2\,\mu_1\,\mathbb{E}\big\{\Lambda^{-1}\big\}}{\sigma^2}\,\frac{1}{t} \quad \text{as} \quad t \to \infty$$

where $\sigma^2 := \mathbb{V}X < \infty$.

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