

# ANALYSIS OF RISK MEASURES FOR REINSURANCE LAYERS

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ABSTRACT. We analyze common risk measures for reinsurance layers defined in terms of lower and upper retentions. In particular, we consider the Value-at-Risk, the variance, the coefficient of variation, the dispersion and the reduction effect. In a first part, we compute some risk measures for a general layer. In a second part, we compare several risk measures among the different layers in a reinsurance chain.

## 1. INTRODUCTION

Let  $\{Y_i; i \geq 1\}$  be a sequence of successive claim sizes consisting of independent and identically distributed random variables generated by the distribution function  $F_Y$  of a nonnegative random variable  $Y$ . Let  $N$  be a nonnegative integer-valued random variable, independent of the  $Y_i$ 's, representing the number of claims occurring in some fixed time interval. We denote by  $Y_1^*, Y_2^*, \dots, Y_N^*$  the order statistics, arranged in increasing order, of the random sample  $Y_1, Y_2, \dots, Y_N$  of successive claim sizes in the time interval.

Reinsurance can be considered as one way of risk sharing. The reinsurance forms all have in common the intention to diminish an excessive number of claims and/or the impact of the large claims. Of course, reinsurance diminishes the volatility in the portfolio as the risk is shared between the first line insurance and the reinsurance. The decision to involve other partners in the risk sharing depends on many factors, some of them have only marginal relation with reinsurance.

A first line insurance will always try to safeguard its position by subscribing itself to a variety of insurance contracts with an equally varied set of (re)insurance companies. As such, the first line insurance itself becomes an insured client by paying a specific premium to a reinsurance company in exchange for a policy covering the reinsured quantity. For the first line insurance company, it obviously does not make sense to sell the entire portfolio to a reinsurance company because it will then lose all premium income from that portfolio. The first line company has to ponder how it wants the claims to be split between itself and the reinsurer.

It is common to distinguish between two types of reinsurance: proportional and nonproportional. Within the area of proportional reinsurance treaties, we have two traditional forms: *quota-share* reinsurance where the reinsurer accepts a proportion  $a \in (0, 1]$  of the claims experience of the total portfolio and *surplus* reinsurance where the reinsured amount is also determined by the value of the insured object as long as it exceeds a retention  $L$ . Within the framework of nonproportional reinsurance treaties, we cite four forms. An *excess-of-loss* reinsurance is determined by a retention  $M$  indicating that the reinsurer covers the part of the claims that overshoots  $M$ . In a *stop-loss* reinsurance contract, the reinsured amount is the part of the insurer's total loss overshooting a retention  $C$ . Note that excess-of-loss and stop-loss reinsurance treaties are equivalent when a single risk comes into play. Furthermore, there are reinsurance forms classified as large claims reinsurance since they are defined in terms of the order statistics of the claim sizes. A first form is *largest claims* reinsurance where the reinsured amount combines the values of

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the  $r$  largest claim sizes in the portfolio. A second and slightly more popular form is *ECOMOR* reinsurance which is defined as an excess-of-loss treaty with the  $(r + 1)$ th largest claim size as random retention. We refer to Teugels [11], Embrechts et al. [3] and Ladoucette and Teugels [6] for asymptotic problems pertaining to ECOMOR as well as to largest claims reinsurance. For an overview of most of the currently employed reinsurance forms with some of their properties, see Rolski et al. [8] and Teugels [12].

In accordance with current practice, quota-share and stop-loss treaties are somewhat popular if the number of claims is large. Also, surplus and excess-of-loss reinsurance are more traditional if the claim sizes are large. Furthermore, reinsurance based on the largest claims is almost never used. This fact is rather surprising if reinsurance is meant to protect the insurer against large claims since, in such a case, it looks almost necessary to use a reinsurance treaty that is based on these largest claims.

Combinations of different forms of treaties are easily constructed. Schmitter [9] combines quota-share and stop-loss treaties such that the reinsured amount is:

$$R_{a,C} := \max \left( 0, a \sum_{i=1}^N Y_i - C \right).$$

Quota-share and excess-of-loss treaties are combined in Centeno [2]. The reinsured amount then has the following expression:

$$R_{a,M} := \sum_{i=1}^N \max(0, aY_i - M).$$

Benktander and Ohlin [1] combine a surplus treaty with an excess-of-loss treaty. The reinsured amount is then:

$$R_{L,M} := \sum_{i=1}^N \max \left( 0, \left( \frac{V_i - L}{V_i} \right) Y_i \mathbb{1}_{\{V_i > L\}} - M \right)$$

where  $V_i$  is the value insured for the  $i$ th claim policy. Here and throughout the paper,  $\mathbb{1}_A$  stands for the indicator function of the event  $A$ .

Such combinations are referred to as *partial reinsurance*. For example, see Steenackers and Goovaerts [10]. Even more popular is the combination of a stop-loss treaty on top of an excess-of-loss treaty. In this case, one has for the reinsured amount:

$$R_{M,C} := \max \left\{ 0, \sum_{i=1}^N \max(0, Y_i - M) - C \right\}.$$

A further generalization is called *drop down excess-of-loss* reinsurance. In this type of setting, the claim size is curtailed at both ends, both of them depending on the order of the claim. The reinsured amount has a form of the kind:

$$R_{M_i, L_i} := \sum_{i=1}^N \min \{ L_i, \max(0, Y_{N-i+1}^* - M_i) \}$$

where the  $M_i$ 's stand for the lower retentions while the  $L_i$ 's determine the drop down upper retentions  $M_i + L_i$ .

Consider a nonnegative random variable  $X$  with distribution function  $F_X$ . As examples,  $X$  may represent an individual claim size  $Y_i$ , the  $i$ th largest claim size  $Y_{N-i+1}^*$  among  $Y_1, \dots, Y_N$  or even the total claim amount  $\sum_{i=1}^N Y_i$ , where  $N$  is the number of claims occurred over a given period of time. Choose real constants  $u \in [0, \infty)$ ,  $v \in (0, \infty]$  and  $a \in (0, 1]$  and define:

$$(1) \quad X_{a,u,v} := a \min \{ v, \max(0, X - u) \}.$$

One may consider  $u$  as a lower retention or deductible,  $u + v$  as an upper retention or limit of coverage and  $a$  as a proportionality factor. Further, it is easy to see that  $X_{a,u,v}$  may be rewritten in the following

more explicit form:

$$X_{a,u,v} = \begin{cases} 0, & X \leq u \\ a(X - u), & u < X \leq u + v \\ av, & X > u + v. \end{cases}$$

From its definition, it immediately follows that  $0 \leq X_{a,u,v} \leq av$  with probability 1, i.e. that  $X_{a,u,v}$  is a nonnegative random variable which is bounded when  $v \neq \infty$ . Moreover, the distribution function  $F_{X_{a,u,v}}$  of  $X_{a,u,v}$  is simply given by:

$$(2) \quad F_{X_{a,u,v}}(x) := \mathbb{P}[X_{a,u,v} \leq x] = \begin{cases} 0, & x < 0 \\ F_X(u + x/a), & 0 \leq x < av \\ 1, & x \geq av. \end{cases}$$

If  $X$  represents an individual claim size  $Y_i$  then summing over  $i = 1, \dots, N$  the form (1) gives a combination of excess-of-loss with an upper limit of coverage and quota-share reinsurance treaties. In the case  $v = \infty$ , one arrives at a quota-share reinsurance if  $u = 0$  whereas one gets a combination of an excess-of-loss with a quota-share treaty if  $u \neq 0$ . When  $X$  is used in (1) as the total claim amount  $\sum_{i=1}^N Y_i$ , we obtain a quota-share on top of a stop-loss treaty with an upper limit of coverage. These reinsurance forms make the classical treaties a bit more flexible by also allowing an upper retention. This seems worthwhile in view of the different situations between first and second line insurance. For a first line insurer,  $u = 0$  while  $v = M < \infty$ . For the first reinsurer however,  $u = M$  while  $v$  may take any positive value. If the first line reinsurer does not shift part of the risk to a second reinsurer then  $v = \infty$ . Otherwise  $v < \infty$  and so on. By not specifying  $u$  and  $v$ , our results apply to any company in a reinsurance chain of this type. Of course, any reinsurer can apply the type of reinsurance of his choice, irrespective of what a former insurer has been doing.

In the following, we are interested in the behavior of some risk measures that are frequently used in practical situations, in particular in reinsurance. Risk measures are useful in evaluating or estimating the risk associated with a random quantity. In the context of (re)insurance, such measures also permit to illustrate and quantify the effect of a premium scheme. Look in Venter [13] where it is advocated to use a variety of risk measures.

In Section 2, we investigate a couple of risk measures for the quantity  $X_{a,u,v}$  defined in (1). We successively deal with the Value-at-Risk (Subsection 2.1), the variance (Subsection 2.2) and the coefficient of variation (Subsection 2.3). Given that (1) is applied to each of  $N$  individual claim sizes, the computation of risk measures for the corresponding sum reinsured is discussed in Section 3. More specifically, we assume a reinsurance chain for the insurer's aggregate claims where each layer is defined in terms of the parameters  $u$  and  $v$ . Then, we compare several risk measures among the different layers, in that way investigating properties of the risk sharing. We consider the variance (Subsection 3.1), the dispersion (Subsection 3.2), the coefficient of variation (Subsection 3.3) and the reduction effect (Subsection 3.4). The conclusion is given in Section 4.

For theoretical and practical results pertaining to the coefficient of variation within the context of (re)insurance, see Mack [7]. For the asymptotic behavior of the sample coefficient of variation as well as for the sample dispersion, see the recent papers by Ladoucette [4] and Ladoucette and Teugels [5].

## 2. RISK MEASURES FOR $X_{a,u,v}$

We investigate some usual risk measures (Value-at-Risk, variance and coefficient of variation) for the random variable  $X_{a,u,v}$  defined by (1).

We start with a couple of relations that will prove useful later on. As  $X_{a,u,v}$  is nonnegative, we can define its moments of order  $k > 0$  by:

$$\tilde{\mu}_k := \mathbb{E}X_{a,u,v}^k = k \int_0^\infty x^{k-1} (1 - F_{X_{a,u,v}}(x)) dx.$$

Recalling (2), it easily follows that:

$$(3) \quad \tilde{\mu}_k = k \int_0^{av} x^{k-1} (1 - F_X(u + x/a)) dx = ka^k \int_u^{u+v} (x-u)^{k-1} (1 - F_X(x)) dx.$$

This quickly leads to the following lower and upper bounds:

$$(av)^k (1 - F_X(u+v)) \leq \tilde{\mu}_k \leq (av)^k (1 - F_X(u)).$$

Since  $X_{a,u,v} \leq av$  with probability 1, we also have  $\mathbb{E}X_{a,u,v}^k \leq av \mathbb{E}X_{a,u,v}^{k-1}$  for any  $k > 1$ , i.e.:

$$(4) \quad \tilde{\mu}_k \leq av \tilde{\mu}_{k-1}.$$

Using (3), we calculate the partial derivative of  $\tilde{\mu}_k$  with respect to  $v$  to be:

$$(5) \quad \frac{\partial \tilde{\mu}_k}{\partial v} = ak(av)^{k-1} (1 - F_X(u+v)).$$

We also deduce the following relation between the partial derivatives of two successive moments:

$$(6) \quad \frac{\partial \tilde{\mu}_{k+1}}{\partial v} = \frac{k+1}{k} av \frac{\partial \tilde{\mu}_k}{\partial v}.$$

**2.1. Value-at-Risk.** We define the *Value-at-Risk* of a random variable  $Z$  with distribution function  $F_Z$  by:

$$\text{VaR}_q(Z) := U_Z(q), \quad q > 1$$

where  $U_Z$  is the *tail quantile function* of  $F_Z$  defined by  $U_Z(y) := \inf\{x : F_Z(x) \geq 1 - 1/y\}$  for  $y > 1$ . If  $Z$  represents a claim size, it is rather clear that when  $q$  is large, this is only defined for the reinsurer that carries the ultimate tail of the claim.

For the quantity  $X_{a,u,v}$ , we are naturally led to the following left-continuous Value-at-Risk function:

$$\text{VaR}_q(X_{a,u,v}) = \begin{cases} 0, & 1 < q \leq U_X^-(u) \\ a(U_X(q) - u), & U_X^-(u) < q \leq U_X^-(u+v) \\ av, & q > U_X^-(u+v) \end{cases}$$

where  $U_X^-$  denotes the left-continuous inverse function of  $U_X$  defined by  $U_X^-(x) := \inf\{y : U_X(y) \geq x\}$  for  $x \geq 0$ .

In addition, note that we have:

$$\text{VaR}_q(X_{a,u,v}) \leq \text{VaR}_q(aX) = a \text{VaR}_q(X) \leq \text{VaR}_q(X), \quad q > 1.$$

**2.2. Variance.** Next, we deal with the *variance*  $\mathbb{V}X_{a,u,v} := \tilde{\mu}_2 - \tilde{\mu}_1^2$  of the random variable  $X_{a,u,v}$ . From (3), it is immediate that:

$$(7) \quad \mathbb{V}X_{a,u,v} = 2 \int_0^{av} x(1 - F_X(u + x/a)) dx - \left( \int_0^{av} (1 - F_X(u + x/a)) dx \right)^2.$$

By using (5), we therefore get:

$$\begin{aligned} \frac{\partial \mathbb{V}X_{a,u,v}}{\partial v} &= \frac{\partial \tilde{\mu}_2}{\partial v} - 2\tilde{\mu}_1 \frac{\partial \tilde{\mu}_1}{\partial v} \\ &= 2a^2v(1 - F_X(u+v)) - 2a(1 - F_X(u+v)) \int_0^{av} (1 - F_X(u + x/a)) dx \\ &= 2a(1 - F_X(u+v)) \int_0^{av} F_X(u + x/a) dx \geq 0. \end{aligned}$$

The partial derivative with respect to  $v$  being nonnegative, the variance is nondecreasing in  $v$  so that:

$$\begin{aligned} \mathbb{V}X_{a,u,v} &\leq 2 \int_0^\infty x(1 - F_X(u + x/a)) dx - \left( \int_0^\infty (1 - F_X(u + x/a)) dx \right)^2 \\ &= 2a^2 \int_u^\infty (y-u)(1 - F_X(y)) dy - a^2 \left( \int_u^\infty (1 - F_X(y)) dy \right)^2 =: g_a(u). \end{aligned}$$

Now, we compute:

$$\begin{aligned}\frac{\partial g_a(u)}{\partial u} &= -2a^2 \int_u^\infty (1 - F_X(y)) dy + 2a^2 (1 - F_X(u)) \int_u^\infty (1 - F_X(y)) dy \\ &= -2a^2 F_X(u) \int_u^\infty (1 - F_X(y)) dy \leq 0.\end{aligned}$$

Thus, the function  $g_a(u)$  is nonincreasing in  $u$  and therefore smaller than the same expression where we put  $u = 0$ , that is  $g_a(0)$ . However, this quantity is equal to  $a^2 \mathbb{V}X$ . Hence, we get the following result for the variance of  $X_{a,u,v}$ :

$$\mathbb{V}X_{a,u,v} \leq a^2 \mathbb{V}X.$$

Since  $a \in (0, 1]$ , we also remark that  $\mathbb{V}X_{a,u,v} \leq \mathbb{V}X$  as it should be.

**2.3. Coefficient of Variation.** Another risk measure that is popular among actuaries is the *coefficient of variation*, a dimensionless relative measure of dispersion around the mean. It is defined for a random variable  $Z$  satisfying  $\mathbb{E}Z \neq 0$  by:

$$\text{CoVar}(Z) := \frac{\sqrt{\mathbb{V}Z}}{\mathbb{E}Z}.$$

Note that  $\text{CoVar}(cZ) = \text{CoVar}(Z)$  for any  $c > 0$ .

For the random variable  $X_{a,u,v}$ , we obtain:

$$\text{CoVar}(X_{a,u,v}) = \sqrt{\frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} - 1}$$

and hence, the coefficient will depend monotonically on the ratio under the square root. Incorporating both retentions into the notation and since  $\text{CoVar}(X_{a,u,v}) = \text{CoVar}(X_{1,u,v})$ , we write  $\text{CoVar}(u, v) := \text{CoVar}(X_{a,u,v})$  for arbitrary  $u$  and  $v$ .

Inequality (4) together with (6) leads to the following:

$$\begin{aligned}\frac{\partial \tilde{\mu}_2}{\partial v \tilde{\mu}_1^2} &= (\tilde{\mu}_1)^{-3} \left( \tilde{\mu}_1 \frac{\partial \tilde{\mu}_2}{\partial v} - 2\tilde{\mu}_2 \frac{\partial \tilde{\mu}_1}{\partial v} \right) \\ &\geq (\tilde{\mu}_1)^{-2} \left( \frac{\partial \tilde{\mu}_2}{\partial v} - 2av \frac{\partial \tilde{\mu}_1}{\partial v} \right) = 0.\end{aligned}$$

The above then shows that  $\text{CoVar}(u, v)$  is nondecreasing in  $v$ . Hence, we get:

$$(8) \quad \text{CoVar}(u, v) \leq \text{CoVar}(u, \infty).$$

The dependence on  $u$  is more intricate. Let us introduce the *retention distribution*:

$$G_{a,u,v}(x) := \begin{cases} \frac{F_X(u+x/a) - F_X(u)}{F_X(u+v) - F_X(u)}, & 0 \leq x < av \\ 1, & x \geq av \end{cases}$$

with moments of order  $k > 0$  given by  $\nu_k := k \int_0^\infty x^{k-1} (1 - G_{a,u,v}(x)) dx$ .

If we abbreviate  $\Delta_X := F_X(u+v) - F_X(u)$ , we then get:

$$\begin{aligned}\Delta_X \nu_k &= k (F_X(u+v) - F_X(u)) \int_0^{av} x^{k-1} \left( 1 - \frac{F_X(u+x/a) - F_X(u)}{F_X(u+v) - F_X(u)} \right) dx \\ &= -k \int_0^{av} x^{k-1} \{1 - F_X(u+v) - (1 - F_X(u+x/a))\} dx \\ &= -(av)^k (1 - F_X(u+v)) + k \int_0^{av} x^{k-1} (1 - F_X(u+x/a)) dx \\ (9) \quad &= \tilde{\mu}_k - (av)^k (1 - F_X(u+v)).\end{aligned}$$

The latter relation is handy in rewriting the partial derivative of  $\text{CoVar}(u, v)$  with respect to  $u$ . Indeed, it follows that:

$$\frac{\partial \tilde{\mu}_1}{\partial u} = - \int_0^{av} \frac{\partial}{\partial u} F_X(u+x/a) dx = -a (F_X(u+v) - F_X(u)) = -a \Delta_X$$

and:

$$\begin{aligned}
\frac{\partial \tilde{\mu}_2}{\partial u} &= -2a \int_0^{av} x d_x F_X(u + x/a) \\
&= \left[ -2ax F_X(u + x/a) \right]_0^{av} + 2a \int_0^{av} F_X(u + x/a) dx \\
&= -2a^2 v F_X(u + v) + 2a \int_0^{av} F_X(u + x/a) dx \\
&= -2a \int_0^{av} (1 - F_X(u + x/a)) dx + 2a^2 v - 2a^2 v F_X(u + v) \\
&= 2a^2 v (1 - F_X(u + v)) - 2a \int_0^{av} (1 - F_X(u + x/a)) dx = -2a \Delta_X \nu_1.
\end{aligned}$$

But then, we have:

$$\tilde{\mu}_1^3 \frac{\partial \tilde{\mu}_2}{\partial u \tilde{\mu}_1^2} = \tilde{\mu}_1 \frac{\partial \tilde{\mu}_2}{\partial u} - 2\tilde{\mu}_2 \frac{\partial \tilde{\mu}_1}{\partial u} = 2a \Delta_X (\tilde{\mu}_2 - \tilde{\mu}_1 \nu_1).$$

Replacing in the last expression the moments  $\tilde{\mu}_k$  by their analogues  $\nu_k$  from (9), we get:

$$\begin{aligned}
\tilde{\mu}_1^3 \frac{\partial \tilde{\mu}_2}{\partial u \tilde{\mu}_1^2} &= 2a \Delta_X \{ \Delta_X \nu_2 + (av)^2 (1 - F_X(u + v)) - \nu_1 (\Delta_X \nu_1 + av (1 - F_X(u + v))) \} \\
&= 2a \Delta_X^2 (\nu_2 - \nu_1^2) + 2a^2 \Delta_X v (av - \nu_1) (1 - F_X(u + v)) \geq 0.
\end{aligned}$$

Indeed, the quantity  $\nu_2 - \nu_1^2$  is the variance of the distribution  $G_{a,u,v}$  while by definition  $\nu_1 \leq av$ . This then shows that the requested partial derivative is nonnegative and hence that  $\text{CoVar}(u, v)$  is also nondecreasing in  $u$ . Consequently,  $\text{CoVar}(u, v)$  is nondecreasing both in  $u$  and  $v$ . In particular, we find that:

$$(10) \quad \text{CoVar}(u, v) \geq \text{CoVar}(0, v).$$

Applying both inequalities (8) and (10), we get the following lower and upper bounds for the coefficient of variation of  $X_{a,u,v}$ :

$$\text{CoVar}(0, v) \leq \text{CoVar}(u, v) \leq \text{CoVar}(u, \infty)$$

comparing the risk measure between different layers for any  $a \in (0, 1]$ . Applying the inequality on the left for  $v = \infty$  and that on the right for  $u = 0$ , we also get the following inequalities:

$$\text{CoVar}(0, v) \leq \text{CoVar}(0, \infty) = \text{CoVar}(X) \leq \text{CoVar}(u, \infty)$$

where the quantity in the middle is the coefficient of variation of the initial risk  $X$ . The latter inequality has been obtained by Mack [7] for  $a = 1$ .

### 3. RISK MEASURES FOR REINSURANCE LAYERS

Taking a combination of a quota-share treaty on top of an excess-of-loss treaty with an upper limit of coverage, the reinsured amount in the time interval is:

$$(11) \quad R_{a,u,v} := \sum_{i=1}^N Y_{i;a,u,v}$$

where the random variable  $Y_{i;a,u,v} := a \min \{v, \max(0, Y_i - u)\}$  is the reinsured part of the original claim size  $Y_i$  defined by (1) for  $i = 1, \dots, N$ .

Using the so-called Wald identities, we easily derive the following expressions for the mean and the variance of the quantity  $R_{a,u,v}$ :

$$\mathbb{E}R_{a,u,v} = \tilde{\mu}_1 \mathbb{E}N$$

$$\mathbb{V}R_{a,u,v} = \mathbb{V}Y_{a,u,v} \mathbb{E}N + \tilde{\mu}_1^2 \mathbb{V}N$$

with  $Y_{a,u,v} := a \min \{v, \max(0, Y - u)\}$  and where  $\tilde{\mu}_1$  and  $\mathbb{V}Y_{a,u,v}$  are respectively given by (3) and (7), replacing  $F_X$  by  $F_Y$  in these formulas.

By way of example and without loosing too much of generality, we assume from now on that there are exactly three partners involved. For the first line, there is a deductible determined by a retention  $u$ . The first reinsurer is responsible for the part of the claim sizes above the level  $u$  and up to the retention  $u + v$ . Finally, a last reinsurer deals with all the claim sizes above the level  $u + v$ . Consequently, we assume that  $F_Y(u)\Delta_Y(1 - F_Y(u + v)) > 0$ . To investigate the effect of reinsurance on the partners involved, let us introduce some notation.

- (1) For the first insurer, the aggregate loss is  $S_L := \sum_{i=1}^N L_i$  where  $L_i$  has the same distribution as the random variable:

$$L := aY \mathbb{1}_{\{Y \leq u\}} + au \mathbb{1}_{\{Y > u\}} = a \min(u, Y).$$

This corresponds to the choice  $u = 0$  and  $v = u$  in the general approach (11). We use the letter  $L$  to point at the lower part of the claims.

- (2) The first reinsurer's aggregate loss is  $S_M := \sum_{i=1}^N M_i$  where  $M_i$  has the same distribution as the random variable:

$$M := a(Y - u) \mathbb{1}_{\{u < Y \leq u + v\}} + av \mathbb{1}_{\{Y > u + v\}} = a \min\{v, \max(0, Y - u)\}.$$

This case corresponds to the general approach (11) with arbitrary  $u$  and  $v$ . The letter  $M$  points to the middle part of the claims.

- (3) Finally, the top layer is carried by the second and last reinsurer. Its aggregate loss is given by  $S_U := \sum_{i=1}^N U_i$  where  $U_i$  has the same distribution as the random variable:

$$U := a(Y - (u + v)) \mathbb{1}_{\{Y > u + v\}} = a \max(0, Y - (u + v)).$$

In (11), we need to replace  $u$  by  $u + v$  while  $v = \infty$ . We use the letter  $U$  to indicate that the upper part of the claims is now at stake.

Notice that the sum of the three quantities  $S_L$ ,  $S_M$  and  $S_U$  adds up to  $R_{a,0,\infty} = a \sum_{i=1}^N Y_i$ . In the case  $a \neq 1$ , the remaining part  $(1 - a) \sum_{i=1}^N Y_i$  of the total claim amount could be considered as taken out, prior to the risk sharing of the part  $a \sum_{i=1}^N Y_i$  by the three partners.

As an additional risk measure, let us also look at the *covariances*. Recall that the covariance of two random variables  $Z_1$  and  $Z_2$  is defined as  $\text{Cov}(Z_1, Z_2) := \mathbb{E}\{Z_1 Z_2\} - \mathbb{E}Z_1 \mathbb{E}Z_2$ . From the definitions of the three variables  $L$ ,  $M$  and  $U$ , it follows that  $\mathbb{E}\{LM\} = au \mathbb{E}M$ ,  $\mathbb{E}\{LU\} = au \mathbb{E}U$  and  $\mathbb{E}\{MU\} = av \mathbb{E}U$ . Therefore, the three covariances  $\text{Cov}(L, M)$ ,  $\text{Cov}(L, U)$  and  $\text{Cov}(M, U)$  are all positive. To see the influence on the three separate aggregate losses, we calculate the covariances between the quantities  $S_L$ ,  $S_M$  and  $S_U$ . An easy calculation learns us that:

$$\mathbb{E}\{S_L S_M\} = \sum_{n=0}^{\infty} \mathbb{P}[N = n] \sum_{i=1}^n \mathbb{E}\{L_i M_i\} + \sum_{n=0}^{\infty} \mathbb{P}[N = n] \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{E}\{L_i M_j\}$$

and hence:

$$(12) \quad \text{Cov}(S_L, S_M) = \text{Cov}(L, M) \mathbb{E}N + \mathbb{E}L \mathbb{E}M \mathbb{V}N.$$

In equation (12), one can replace the pair  $(L, M)$  by  $(L, U)$  and  $(M, U)$  without any problem. Using the expressions for the covariances between the variables  $L$ ,  $M$  and  $U$ , one finds the expressions:

$$\text{Cov}(S_L, S_M) = \mathbb{E}L \mathbb{E}M (\mathbb{V}N - \mathbb{E}N) + au \mathbb{E}M \mathbb{E}N$$

$$\text{Cov}(S_L, S_U) = \mathbb{E}L \mathbb{E}U (\mathbb{V}N - \mathbb{E}N) + au \mathbb{E}U \mathbb{E}N$$

$$\text{Cov}(S_M, S_U) = \mathbb{E}M \mathbb{E}U (\mathbb{V}N - \mathbb{E}N) + av \mathbb{E}U \mathbb{E}N.$$

In the following, we compare some risk measures for the different partners in the chain, namely the variance, the dispersion, the coefficient of variation and the reduction effect. Let  $A$  and  $B$  be any two distinct letters from the set  $\{L, M, U\}$ .

**3.1. Variance.** First, we deal with the variance. If one uses the variance as a risk measure then one should compare the variance of a combined risk with the sum of the individual variances. It is clear that:

$$\mathbb{V}\{S_A + S_B\} = \mathbb{V}\{A + B\} \mathbb{E}N + (\mathbb{E}A + \mathbb{E}B)^2 \mathbb{V}N.$$

Since (12) learns us that  $\text{Cov}(S_A, S_B) > 0$ , we get by a simple calculation:

$$(13) \quad \mathbb{V}\{S_A + S_B\} = \mathbb{V}S_A + \mathbb{V}S_B + 2\text{Cov}(S_A, S_B) > \mathbb{V}S_A + \mathbb{V}S_B.$$

A similar relation holds between the two other pairs. Hence, there is an increase in variance if one combines different layers in the reinsurance chain.

**3.2. Dispersion.** The situation is different for the *dispersion*. Recall that the dispersion of a random variable  $Z$  satisfying  $\mathbb{E}Z \neq 0$  is defined by:

$$D(Z) := \frac{\mathbb{V}Z}{\mathbb{E}Z}$$

and compares the volatility of  $Z$  with that of the Poisson case.

The correlation coefficient between  $S_A$  and  $S_B$  is at most 1 in absolute value. Hence, we see that  $\text{Cov}(S_A, S_B) < \sqrt{\mathbb{V}S_A \mathbb{V}S_B}$  and it follows from (13) that:

$$(14) \quad \sqrt{\mathbb{V}\{S_A + S_B\}} < \sqrt{\mathbb{V}S_A} + \sqrt{\mathbb{V}S_B}.$$

Using (14), we calculate:

$$\begin{aligned} D(S_A + S_B) &= \frac{\mathbb{V}\{S_A + S_B\}}{\mathbb{E}\{S_A + S_B\}} \\ &< \frac{\mathbb{V}S_A + \mathbb{V}S_B + 2\sqrt{\mathbb{V}S_A \mathbb{V}S_B}}{\mathbb{E}\{S_A + S_B\}} \\ &= D(S_A) + D(S_B) - \left( \frac{\mathbb{V}S_A \mathbb{E}S_B}{\mathbb{E}S_A \mathbb{E}\{S_A + S_B\}} + \frac{\mathbb{V}S_B \mathbb{E}S_A}{\mathbb{E}S_B \mathbb{E}\{S_A + S_B\}} - 2 \frac{\sqrt{\mathbb{V}S_A \mathbb{V}S_B}}{\mathbb{E}\{S_A + S_B\}} \right) \\ &= D(S_A) + D(S_B) - \left( \frac{\mathbb{V}S_A (\mathbb{E}S_B)^2 + \mathbb{V}S_B (\mathbb{E}S_A)^2 - 2\sqrt{\mathbb{V}S_A \mathbb{V}S_B} \mathbb{E}S_B \mathbb{E}S_A}{\mathbb{E}S_A \mathbb{E}S_B \mathbb{E}\{S_A + S_B\}} \right) \\ &\leq D(S_A) + D(S_B). \end{aligned}$$

Therefore, the dispersion decreases by combining different layers.

**3.3. Coefficient of Variation.** Let us look at the coefficient of variation of  $S_A$ . From general principles, we derive:

$$\begin{aligned} \frac{\mathbb{V}S_A}{(\mathbb{E}S_A)^2} &= \frac{\mathbb{V}A \mathbb{E}N + (\mathbb{E}A)^2 \mathbb{V}N}{(\mathbb{E}A \mathbb{E}N)^2} \\ &= \frac{\mathbb{V}A}{(\mathbb{E}A)^2 \mathbb{E}N} + \frac{\mathbb{V}N}{(\mathbb{E}N)^2}. \end{aligned}$$

Hence, we obtain the interesting formula:

$$\text{CoVar}^2(S_A) = \frac{\text{CoVar}^2(A)}{\mathbb{E}N} + \text{CoVar}^2(N).$$

The second term on the right depends on the number of claims and is the same for all partners involved. Also the factor  $(\mathbb{E}N)^{-1}$  is the same. Therefore, the differences in the risk measure solely depend on the coefficient of variation for the retained risk. However, the comparison of this risk measure or the quantity  $\tilde{\mu}_2/\tilde{\mu}_1^2$  for the different values of  $u$  and  $v$  is by no means simple.

Moreover, using (14), we easily deduce:

$$\text{CoVar}(S_A + S_B) = \frac{\sqrt{\mathbb{V}\{S_A + S_B\}}}{\mathbb{E}\{S_A + S_B\}} < \frac{\sqrt{\mathbb{V}S_A}}{\mathbb{E}S_A} + \frac{\sqrt{\mathbb{V}S_B}}{\mathbb{E}S_B} = \text{CoVar}(S_A) + \text{CoVar}(S_B).$$

Thus, by combining different layers in the reinsurance chain, the coefficient of variation decreases.

**3.4. Reduction Effect.** As another risk measure, we finally look at the *reduction effect*. There are at least two ways of writing out a reduction effect depending on the partner one is looking at. For the first insurer however, it does hardly matter how the first reinsurance parameter  $v$  is decided. Therefore, it seems natural to calculate the reduction effect by the quantity:

$$r_L := \frac{\mathbb{E}S_L}{\mathbb{E}\{S_L + S_M + S_U\}}.$$

However, this is easily calculated since:

$$r_L = \frac{\mathbb{E}L}{a\mathbb{E}Y} = \frac{1}{\mathbb{E}Y} \int_0^u (1 - F_Y(x)) dx.$$

The latter quantity is well known from renewal theory where it is called the *equilibrium distribution* of  $F_Y$ . Properties of this distribution can be found in standard treatises of renewal theory.

For the first reinsurer, the natural reduction effect is:

$$r_M := \frac{\mathbb{E}S_M}{\mathbb{E}\{S_M + S_U\}}$$

as only the reduction with respect to the last reinsurer needs to be taken into account. We then obtain:

$$r_M = \frac{\mathbb{E}M}{\mathbb{E}M + \mathbb{E}U} = \frac{\int_0^v (1 - F_Y(u+x)) dx}{\int_0^\infty (1 - F_Y(u+x)) dx} =: F(v).$$

Note that the quantity on the right is indeed a proper distribution which for every fixed value of  $u$  can be considered as an equilibrium distribution.

#### 4. CONCLUSION

In this paper, we have been concerned with the analysis of common risk measures for reinsurance layers, namely the Value-at-Risk, the variance, the coefficient of variation, the dispersion and the reduction effect. We have derived some risk measures for a general layer. Next, we have investigated the effect of reinsurance on the partners involved in a reinsurance chain by comparing several risk measures for the remaining risk of each partner.

We point out that the results of Section 2 are useful ingredients for the analysis of a combination of arbitrary drop down excess-of-loss with quota-share reinsurance. On this subject, a forthcoming paper will be devoted to the study of the reinsured amount in such a reinsurance form on an ordered claim size. In other words, we will take  $X$  in (1) to be the  $i$ th largest claim size  $Y_{N(t)-i+1}^*$  in the randomly indexed sample  $Y_1, \dots, Y_{N(t)}$ , the parameters  $u$  and  $v$  being replaced by deterministic functions both depending on time  $t$  and rank  $i$ . Results pertaining to exact and asymptotic properties of this quantity will then be derived.

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