The practical pricing of Excess of Loss treaties: actuarial, financial, economic and commercial aspects

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Abstract. In this paper, we investigate actuarial, financial, economic and commercial aspects of the pricing of an excess of loss treaty. The flexible model we propose allows the calculation of premium rates for all kinds of excess of loss treaties, even with specific clauses. We give a description of the methodology and we illustrate it with various numerical examples. The flexibility of the model easily allows for sensitivity analyses.

Keywords: Panjer's algorithm, inflation, payment pattern, stability clause, capital allocation, cost of capital, cash flow model, premium principle.

1 Introduction

In this paper we propose a comprehensive methodology for the practical pricing of excess of loss treaties. We will work within the collective risk model. However all the material described in the present paper may be extended within the individual risk model (see e.g. Klugmann et al. (1998) for a comprehensive description of these risk models).

Let X be the random variable representing claim amounts with cumulative density function (cdf) $F_X(x)$ and probability density function (pdf) $f_X(x)$.

Let N be the random variable representing claim numbers with pdf $\mathbb{P}[N=n]=p(n)$.

Let P be the priority and L be the limit of the excess of loss treaty.

It is well known that such a treaty covers the part of each claim in excess of P with a maximum liability L. The loss for the reinsurer writes:

$$Y_i = \min(L, \max(0, X_i - P)).$$

Within the collective risk model, S denotes the aggregate claims the reinsurer is liable for:

$$S = Y_1 + \cdots + Y_N.$$

However the treaty is often assorted with clauses limiting the aggregate liability of the reinsurer and / or letting the reinsurance premium be random. In these cases it is well known that the whole distribution of S is needed even to find the pure premium associated to the treaty. We will review these clauses in section 2 (clauses limiting the aggregate liability of the reinsurer) and in section 8 (clauses letting the reinsurance premium be random).

In the classical European actuarial literature, premiums are calculated according to the so-called premium principles. Let S be a risk, described by its cdf F_S. A premium principle associates a premium $H(S)$ to a risk S. Some examples of premium principles are

- expected value premium principle:
  $H(S) = (1+\alpha)\mathbb{E}S, \quad \alpha \geq 0,$
- standard deviation premium principle:
  $H(S) = \mathbb{E}S + \gamma\sqrt{\mathbb{V}arS}, \quad \gamma \geq 0,$
- PH-transform premium principle:
  $H(S) = \int_0^\infty (1-F_S(s))^\rho ds, \quad 0 < \rho \leq 1.$

The safety loading is defined as $H(S) - \mathbb{E}S$, that is the loaded premium minus the pure premium. Rolski et al. (1999) write "the safety loading has to take care not only of administrative costs from handling the portfolio, but also of the necessary gain that the company wants to make ultimately". With such a viewpoint, we are confronted with some practical difficulties:

- The loading of the pure premium as provided by the different premium principles covers various economic factors: financial revenue on capital as well as on loss reserves, risk associated to the full underwriting process, shareholder's profit (i.e. shareholder's remuneration for allocated capital), management expenses, cost of retrocession, ... All those factors must be captured by the parameter α, γ or ρ which is not an easy task.
- It is also obvious that some of the above-mentioned factors (e.g. risk) are to be handled taken as a whole for a firm's portfolio of business and not locally for each piece of business.

On the other hand, North American literature has been relying since a long time on financial approaches for the pricing of non life business. In particular, cash flow models are advocated by several authors (see e.g. D'Arcy and Dyer (1997), Cummins (1990), ...).

The general methodology we will present in this paper for the pricing of excess of loss treaties will be based on cash flow models. Indeed the payments of the reinsurance premium and of losses do not take place at the same moment. This is the classical inversion of the economic cycle for (re)insurers: premiums are collected in a first time and losses, if any, are

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paid later. This is more relevant for long-tail business (e.g. motor third party liability) than for short tail business (e.g. fire). Even more so for long-tail excess of loss reinsurance where the claims may be paid long after the premium instalment. It means that loss payments and loss reserves have to be forecast. We will show in section 3 how to build a model that will enable us to handle the evolution of the reserves and paid losses.

As we observe that cash flows will be paid at different times in the life of an excess of loss treaty, we will present a cash flow model in section 4 that will allow us to make coherent decisions when the future cash flows have been estimated. These techniques are comparable to the classical "investment-decision techniques of financial theory" and are widely accepted especially by North American actuaries. A treaty will be correctly priced when the shareholders requirements are at least fulfilled.

Section 5 will present in detail all the necessary elements for the commercial pricing within the cash flow model.

Section 6 will be devoted to the problem of allocating capital to a specific treaty. Section 7 will provide a short comparison between the European premium principles and the financial economic pricing approach of North American actuaries.

Section 9 will present a numerical application and section 10 will provide various sensitivity analyses. The conclusion will be given in section 11.

2 Clauses limiting the aggregate liability

From now on we adopt the classical convention that treaties are yearly based, which is common practice.

Two clauses usually limit the liability of the reinsurer in an excess of loss treaty. On the one hand the annual aggregate limit (Aal) is the maximal aggregate loss the reinsurer will pay. On the other hand the annual aggregate deductible (Aad) is a deductible on the aggregate loss of the reinsurer. Both annual clauses may coexist. In such a case the loss of the reinsurer writes:

$$S^{Clau} = \min(Aal, \max(0, S - Aad)).$$

Note that without an annual aggregate deductible , we have $Aad = 0$ and without an annual aggregate limit we have $Aal \to \infty$.

Even the determination of the pure premium $\mathbb{E}S^{Clau}$ requires the knowledge of the whole distribution of S.

Fortunately, in most practical cases, it is not too difficult to obtain the distribution of S.

If N belongs to the Panjer's class of counting distribution, i.e. the class of distributions for which the following recursion holds true:

$$\frac{p(n)}{p(n-1)} = a + \frac{b}{n}, \quad n = 1, 2, \ldots$$

then we are able to apply Panjer's algorithm (see Panjer (1981)) in order to find the aggregate claims distribution S:

$$\begin{aligned} f_S(0) &= \psi_N(f_X(0)), \\ f_S(x) &= \frac{1}{1 - af_X(0)} \sum_{i=1}^{x} (a + b\frac{i}{x}) f_X(i) f_S(x-i), \\ & \qquad x = 1, 2, \ldots \end{aligned}$$

where $\psi_N(u) = \mathbb{E}[u^N]$ is the probability generating function of N. Let us recall that there are three members within Panjer's class: Poisson ($a = 0$), Negative Binomial ($0 < a < 1$) and Binomial ($a < 0$).

The Poisson and Negative Binomial distributions are classicaly used by actuaries to model the number of claims. Note that one might work with more complicated counting distributions for which there also exist recursions giving the aggregate claims distribution. The methodology we propose is therefore easy to extend to these models. As a matter of fact, it is also easy to extend within the individual risk model.

As we observe, the recursion is valid for lattice distributions. However the actuary classically fits parametric distributions for the claim amounts, e.g. a Pareto distribution for large claims, as is the case in excess of loss reinsurance. It is therefore necessary to find a discretization of the parametric distribution. There exist several methods in the literature, e.g. the rounding method (see Gerber and Jones (1976)), the local moment matching method (see Gerber (1982)), the minimization of the Kolmogorov distance (see Walhin and Paris (1998)), $\ldots$. Let us choose a span h. The rounding method is the easiest one. It simply accumulates the original mass around the mass points of the lattice distribution (X_{dis}):

$$\begin{aligned} f_{X_{dis}}(0) &= F_X(\frac{h}{2} - 0), \\ f_{X_{dis}}(xh) &= F_X(xh + \frac{h}{2} - 0) - F_X(xh - \frac{h}{2} - 0), \\ & \qquad x = 1, 2, \ldots \end{aligned}$$

Unfortunately the rounding method does not preserve even the first moment of the original distribution. Therefore the local moment matching method was proposed. This method is able to preserve the first moments. However unless for the case where the first moment is conserved, it is not guaranteed that the probabilities of the lattice distribution are positive, which is not acceptable. The minimization of the Kolmogorov distance authorizes the conservation of moments while minimizing the Kolmogorov distance between the original distribution and the lattice distribution. However this method requires a minimization with a huge number of variables, which can make it difficult to use in practice. When the span h is small the quality of the rounding method and local moment matching with one moment are about the same. Therefore we will work with the easiest method in our numerical applications.

Note that the smaller h the better the precision of the discretization however the longer the computing time due to the use of Panjer's algorithm. We will make some sensitivity analysis on the span h.

3 Claims development

3.1 Inflation

In this subsection we will study the influence of inflation and superimposed inflation on the distribution of paid losses.

Let us assume that the distribution of X and N have been estimated, possibly based on past data. Let X_i be a realization of the random variable X. The loss X_i is assumed to be paid in one instalment at the same time as the premium inception. Clearly in long-tail business this is not realistic and we therefore propose a model for the claims development.

We will assume that the payments of the claims occur at times $t_1, t_2, \ldots, t_n$ according to a given claims payment pattern: $c(t_1), \ldots, c(t_n)$ where t_n is the largest time up to which the losses are completely paid. We will make the assumption that $t_0 = 0$, i.e. t_0 is the origin of time in our model. The claims payment pattern is supposed to be estimated on past data and adjusted for potential changes in the future, e.g. due to changes in legislation or change in the claims management.

Moreover the future payments will undergo future inflation. Indeed the losses X_i are assumed not to include any future inflation. Let us define the corresponding index: $inf(t_0), \ldots, inf(t_n)$. The future payments for a loss X_i then write:

$$X_i(t_j) = c(t_j) X_i \frac{inf(t_j)}{inf(t_0)}, \quad j = 1, \ldots, n.$$

As we are interested in high losses, it is commonly observed on the market that this category of losses undergoes a higher inflation than the classical inflation. One speaks of the superimposed inflation. For the future payments, it is then more adequate to use another index, including inflation and superimposed inflation: $supinf(t_0), \ldots, supinf(t_n)$. The future payments for a loss X_i then write:

$$X_i(t_j) = c(t_j) X_i \frac{supinf(t_j)}{supinf(t_0)}, \quad j = 1, \ldots, n.$$

It is also interesting to write the cumulative paid losses:

$$X_i^{\Sigma}(t_j) = \sum_{k=1}^{j} X_i(t_k), \quad j = 1, \ldots, n. \qquad (1)$$

The evolution of the cumulative paid loss for the reinsurer then writes:

$$X_i^{\Sigma Re}(t_j) = \min(L, \max(0, X_i^{\Sigma}(t_j) - P)), \quad j = 1, \ldots, n.$$

In an ideal situation the claims manager is able to calculate exact reserves for each loss:

$$R_i(t_j) = X_i^{\Sigma}(t_n) - X_i^{\Sigma}(t_j), \quad j = 1, \ldots, n.$$

However there may be systematic deviations from these exact reserves. Let us assume that we have observed a pattern of deviation of reservation (overstatement or understatement) : $d(t_1), \ldots, d(t_n)$ where $d(t_j) = 100\%$ if there is no deviation of reservation at time t_j. The incurred loss (paid plus outstanding) may now be written:

$$I_i(t_j) = X_i^{\Sigma}(t_j) + d(t_j) R_i(t_j), \quad j = 1, \ldots n. \qquad (2)$$

From the evolution of this incurred loss, it is now possible to derive the evolution of the incurred loss for the excess of loss reinsurer:

$$I_i^{Re}(t_j) = \min(L, \max(0, I_i(t_j) - P)), \quad j = 1, \ldots, n.$$

3.2 Stability clause

If the priority (P) of the treaty is fixed, the reinsurer will take all future inflation during the development of the claim for his own account. Indeed once the loss exceeds the priority, all future increases due to inflation are borne by the reinsurer only. In order to protect themselves against this kind of possible moral hasard, reinsurers have introduced the stability clause. With this clause the reinsurer is willing to optimally share the future inflation between the ceding company and himself. There are several variants of the stability clause (see e.g. Gerathewohl (1980) for details). In this paper, and in particular in the numerical applications of section 8, we will work with the so-called "date of payment" stability clause. When this clause is applied, the priority and/or the limit of the treaty are indexed each year with the following ratio

$$ratio = \frac{\text{sum of actual payments}}{\text{sum of adjusted payments}},$$

where adjusted payments means that each payment is discounted to the inception of the treaty with use of a conventional index, let us say inf. With our notations, the ratio writes:

$$ratio(t_j) = \frac{\sum_{k=1}^{j} X_i(t_k)}{\sum_{k=1}^{j} X_i(t_k) \frac{inf(t_0)}{inf(t_k)}}, \quad j = 1, \ldots, n.$$

Often there is a margin, i.e. the payments will be adjusted only if inf shows an evolution larger than the margin. Let us assume that the margin is ω. With our notations we find

$$\begin{aligned} \nu(t_k) &= 1 \quad \text{if} \quad \frac{inf(t_k)}{inf(t_0)} \leq 1 + \omega, \\ &= \frac{inf(t_0)}{inf(t_k)} \quad \text{if} \quad \frac{inf(t_k)}{inf(t_0)} > 1 + \omega, \\ ratio(t_j) &= \frac{\sum_{k=1}^{j} X_i(t_k)}{\sum_{k=1}^{j} X_i(t_k) \nu(t_k)}, \quad j = 1, \ldots, n. \end{aligned}$$

In other cases there might be a severe inflation clause (SIC). The SIC follows the same principle as the margin but only the excess of inflation above the margin is taken into account. We find:

$$\begin{aligned}\nu(t_k) &= 1 \quad \text{if} \quad \frac{inf(t_k)}{inf(t_0)} \leq 1+\omega, \\ &= \frac{inf(t_0)(1+\omega)}{inf(t_k)} \quad \text{if} \quad \frac{inf(t_k)}{inf(t_0)} > 1+\omega, \\ ratio(t_j) &= \frac{\sum_{k=1}^{j} X_i(t_k)}{\sum_{k=1}^{j} X_i(t_k)\nu(t_k)}, \quad j=1,\dots,n.\end{aligned}$$

In case of a margin, the influence of the stability clause is reduced and the reduction is even more important when there is a SIC.

If the same claims payment pattern is used for each and every realization of X, it is easy to note that the correction terms $(ratio(t_1), ratio(t_2), \dots, ratio(t_n))$ will be the same for each and every loss.

It means that the priority and the limit of the treaty will change in the future according to the clause. We will thus have future priorities and limits: $P(t_1), P(t_2), \dots, P(t_n)$ and $L(t_1), L(t_2), \dots, L(t_n)$ instead of singles P and L with

$$P(t_j) = ratio(t_j)P, \quad j=1,\dots,n, \tag{3}$$
$$L(t_j) = ratio(t_j)L, \quad j=1,\dots,n. \tag{4}$$

Note that, based on contractual terms, or on common market practice , the stability clause may be applied based on incurred losses instead of on paid losses. In this case the correction term writes

$$ratio(t_j) = \frac{d(t_j)R_i(t_j) + \sum_{k=1}^{j} X_i(t_j)}{d(t_j)R_i(t_j)\nu_j + \sum_{k=1}^{j} X_i(t_k)\nu_k}, \quad j=1,\dots,n.$$

The evolution of the cumulative paid loss and of the incurred loss for the reinsurer now writes :

$$\begin{aligned}X_i^{\Sigma Re}(t_j) &= \min(L(t_j), \max(0, X_i^{\Sigma}(t_j) - P(t_j))), \\ &\qquad j=1,\dots,n, \\ I_i^{Re}(t_j) &= \min(L(t_j), \max(0, I_i(t_j) - P(t_j))), \\ &\qquad j=1,\dots,n.\end{aligned}$$

When the claim is finally settled, both situations lead to the same repartition of the loss between the insurer and the reinsurer. The only difference is in the evolution of the cash flows. Note that the stability clause may provide that only the priority, or only the limit is considered for the application of the clause. Numerical examples will be given in sections 9 and 10.

3.3 Interests sharing clause

When the claims development is long, it is expected that legal interests will have to be paid. The longer the claims development, the higher the legal interests. Once again for moral hasard reasons it may be tempting from the reinsurer's point of view to share the legal interests proportionnally between the cedant and the reinsurer. This is the aim of the interests sharing clause which is common practice, e.g. in Belgium.

The interests sharing clause states that the legal interests have to be shared between the ceding company and the reinsurer according to the pro rata liability of the reinsurer in the total liability of the loss excluding the legal interests. This means that the legal interests have to be excluded from the incurred loss before the application of the treaty. Afterwards, they are divided between the ceding company and the reinsurer in accordance with the pro rata of the liability of both parties in the principal of the loss. Let us assume that on average a proportion δ of the incurred loss represents the interests. Note that it is reasonable to assume that this proportion is a function of the loss. However, in practice, it is extremely difficult to estimate the average proportion of the legal interests in such a way that it does not seem necessary to assume a varying proportion. Nevertheless it is possible to work within an extended model. If $I_i(t_j)$ is the incurred loss, the part of the principal is $(1-\delta)I_i(t_j)$. The liability of the reinsurer in the principal is

$$\min(L(t_j), \max(0, (1-\delta)I_i(t_j) - P(t_j)), \quad j=1,\dots,n. \tag{5}$$

The liability of the reinsurer in the legal interests is

$$\delta I_i(t_j)\frac{\min(L(t_j), \max(0, (1-\delta)I_i(t_j) - P(t_j))}{(1-\delta)I_i(t_j)}, \quad j=1,\dots,n.$$

So now the incurred loss for the reinsurer writes:

$$I_i^{Re}(t_j) = \frac{1}{1-\delta}\min(L(t_j), \max(0, (1-\delta)I_i(t_j) - P(t_j))), \quad j=1,\dots,n. \tag{6}$$

Similarly we obtain the cumulative paid loss of the reinsurer:

$$X_i^{\Sigma Re}(t_j) = \frac{1}{1-\delta}\min(L(t_j), \max(0, (1-\delta)X_i^{\Sigma}(t_j) - P(t_j))), \quad j=1,\dots,n. \tag{7}$$

Numerical examples will be given in sections 9 and 10.

3.4 Aggregate liability of the reinsurer

Our aim is to provide a prediction of the average paid losses and incurred losses of the reinsurer after each year of development. We then have to compute the aggregate losses (paid and incurred) of the reinsurer after each year of development. This is easily done with Panjer's algorithm. However one should be aware of the fact that neither $I^{Re}(t_j)$, $j=1,\dots,n$ nor $X^{\Sigma Re}(t_j)$, $j=1,\dots,n$ are lattice distributions. It is therefore necessary to rearithmetize these distributions. This however is easily done e.g. with the rounding method. Let us define

$S_{X^{\Sigma Re}}(t_j)$, $j = 1, \dots, n$ the aggregate cumulative payments at time t_j and $S_{I^{Re}}(t_j)$, $j = 1, \dots, n$ the aggregate incurred losses at time t_j:

$$\begin{aligned} S_{X^{\Sigma Re}}(t_j) &= \min(Aal, \max(0, \sum_{i=1}^{N} X_i^{\Sigma Re}(t_j) - Aad)), \\ &\qquad j = 1, \dots, n, \\ S_{I^{Re}}(t_j) &= \min(Aal, \max(0, \sum_{i=1}^{N} I_i^{Re}(t_j) - Aad)), \\ &\qquad j = 1, \dots, n. \end{aligned}$$

The incremental payments are

$$\begin{aligned} Paid(t_1) &= S_{X^{\Sigma Re}}(t_1), \\ Paid(t_j) &= S_{X^{\Sigma Re}}(t_j) - S_{X^{\Sigma Re}}(t_{j-1}), \quad j = 2, \dots, n, \end{aligned}$$

and the loss reserves are

$$Reserve(t_j) = S_{I^{Re}}(t_j) - S_{X^{\Sigma Re}}(t_j), \quad j = 1, \dots, n.$$

This situation is the one where the reinsurer follows the information given by the cedant. Another situation might be that the reinsurer books the ultimate loss in such a way that he avoids overstatement and / or understatement of the ceding company's reserves. In this case the loss reserves write:

$$Reserve(t_j) = S_{X^{\Sigma Re}}(t_n) - S_{X^{\Sigma Re}}(t_j), \quad j = 1, \dots, n.$$

We are now able to obtain the average aggregate payments and average aggregate reserves for the reinsurer: $\mathbb{E}Paid(j)$ and $\mathbb{E}Reserve(j)$, $j = 1, \dots, n$.
Note that if there is no annual aggregate limit and no annual aggregate deductible , then the average aggregate payments and reserves are more easily computed by

$$\begin{aligned} \mathbb{E}Paid(t_1) &= \mathbb{E}N\mathbb{E}X^{\Sigma Re}(t_1), \\ \mathbb{E}Paid(t_j) &= \mathbb{E}N\mathbb{E}X^{\Sigma Re}(t_j) - \mathbb{E}N\mathbb{E}X^{\Sigma Re}(t_{j-1}), \\ &\qquad j = 2, \dots, n, \\ \mathbb{E}Reserve(t_j) &= \mathbb{E}N\mathbb{E}I^{Re}(t_j) - \mathbb{E}N\mathbb{E}X^{\Sigma Re}(t_j), \\ &\qquad j = 1, \dots, n. \end{aligned}$$

Observe that $\mathbb{E}S_{X^{\Sigma Re}}(t_n)$ is in fact the technical premium (TP) for the treaty:

$$TP = \mathbb{E}S_{X^{\Sigma Re}}(t_n) = \mathbb{E}S_{I^{Re}}(t_n).$$

It is simply the sum of the future average paid losses. We immediately note that this is a sum of cash flows arising at different moments in the future. We then feel uncomfortable when summing these cash flows.
Numerical examples will be given in section 10.

4 A cash flow model

We saw in the previous section that the losses are not paid in one instalment. There exist thus future cash flows we have to deal with. The aim of this section is to present a cash flow model with this respect.
When a reinsurer wants to write some business he has to provide a solvency margin, or some allocated capital: C. Let us assume that the return the shareholders demand from this capital is *coc*. We call *coc* the cost of capital. It can be derived e.g. via the CAPM (Capital Asset Pricing Model, see e.g. Brealey and Myers (2000)) where $coc = r_F + \beta P_R$. r_F is the risk-free rate and P_R is the risk premium of the market. β measures the systematic risk, i.e. market sensitivity, associated to the investment.
Classically we will say that the business is worth the value if the net present value of all future cash flows, including capital allocation and release, is positive. A null value implies that the requirements of the shareholders are just fulfilled. A positive value implies some creation of value for the shareholders. In the latter case we have the following inequality:

$$0 < \sum_{j=0}^{n} \frac{CF(t_j)}{(1 + coc)^{t_j}}.$$

We will use the cash flow model in this way and say that a treaty is acceptable if the net present value of all future cash flows, including the variations in allocated capital, is positive. Let us note that if the firm is not financed exclusively through equity capital but also through some debt or hybrid capital, *coc* becomes a weighted average cost of capital (see e.g. Brealey and Myers (2000) for details). This however is obviously not very important for insurers and reinsurers which are essentially financed through equity capital.

5 Definition of the cash flows

From now on, in order to simplify our notations, we assume that all cash flows arise at equally spaced times. As we work on annual basis, we logically assume the interspace as being 1 year.
First we have all the cash flows related to losses. We will assume that these cash flows occur in the middle of each year, i.e. at times $t_j = j - 0.5$, $j = 1, \dots n$. This is a reasonable assumption when the payments are uniformly spread over the year. We have three types of cash flows related to losses:

- paid losses: $PL(j - 0.5)$, $j = 1, \dots, n$:

$$PL(j - 0.5) = -\mathbb{E}Paid(t_j).$$

- variation of the loss reserve: $VR(j - 0.5)$, $j = 1, \dots, n$:

$$\begin{aligned} RES(j - 0.5) &= \mathbb{E}Reserve(t_j), \\ VR(0.5) &= -RES(0.5), \\ VR(j - 0.5) &= RES(j - 1.5) - RES(j - 0.5), \end{aligned}$$

- return on reserve: $IR(j - 0.5)$, $j = 1, \dots, n$:

$$\begin{aligned} IR(0.5) &= 0, \\ IR(j - 0.5) &= rRES(j - 1.5). \end{aligned}$$

We logically assume that interests on the reserves are paid with a one year delay.

We can now define the aggregate cash flow at the middle of the year:

$$CF(j-0.5) = \\ PL(j-0.5) + VR(j-0.5) + IR(j-0.5), \\ j = 1, \dots, n.$$

Remember that we defined the technical premium of the treaty as

$$TP = \mathbb{E} S_{X^{\Sigma Re}}(t_n).$$

Within the cash flow model, this definition is not very satisfactory. Indeed

$$\mathbb{E} S_{X^{\Sigma Re}}(t_n) = \sum_{j=1}^{n} \mathbb{E} S_{X^{Re}}(t_j),$$

i.e. it is the sum of non discounted cash flows.
Assume that the premium can be invested in assets yielding a return r. This return is the one we obtain on the loss reserves. We then obtain the discounted technical premium as

$$DTP = \sum_{j=1}^{n} \frac{PL(j-0.5)}{(1+r)^{j-0.5}},$$

that is a premium, obtained at time 0, regardless of the risk associated to deviations either in the paid losses or on the return on the loss reserves.
It is easy to show that the DTP is also given by

$$DTP = \sum_{j=1}^{n} \frac{CF(j-0.5)}{(1+r)^{j-0.5}}.$$

The DTP is not influenced by overstatement or understatement of loss reserves. The commercial pricing, however, will take this fact into account as the variation of reserve may be seen as variation of capital. When evaluating the commercial premium we will make use of the following discounted cash flows:

$$TFP = \sum_{j=1}^{n} \frac{CF(j-0.5)}{(1+coc)^{j-0.5}}.$$

We call the latter the technico-financial premium (TFP).

We will assume that all the other cash flows occur at the beginning of the year: $t_j = j$, $j = 0, 1, \dots, n$. These cash flows are:

- commercial premium ($CP(j)$).
  The premium may be thought to be incepted at time 0. This is not always the case. Often there is a minimum and deposit premium (this concept will be defined in section 9) at time 0. The balance is paid at time 1. We do not take into account (but it is not difficult to do so) the fact that the minimum deposit premium is often paid in different instalments (one quarter every three months or one half every six months). Moreover we will see in section 8 that premium adjustments may be necessary. So premium cash flows at times other than 0 and 1 are not excluded.
- brokerage ($B(j)$).
  Brokerage, if any, is classically a percentage of the commercial premium. It will thus be deducted at times premiums are paid.
- retrocession ($R(j)$).
  Cost of retrocession, if any, is not the premium paid to the retrocessionnaires but rather the expected value of this premium minus the aggregate loss paid by the retrocessionnaire. A possible modelization is a percentage of the commercial premium minus a fraction of the paid losses. The first percentage is the classical rate demanded by the retrocessionnaire on commercial premiums. The latter fraction represents the share of the average claims the retrocessionnaire will pay.
- administrative expenses ($AE(j)$).
  Administrative expenses may be of two types: fixed expenses and proportional expenses. The fixed expenses represent the fixed costs of the reinsurer (including the fixed costs of the priced treaty) whereas the proportional costs represent the costs directly associated with the management of the treaty. We assume that these proportional expenses are based on the paid losses (note that this is just an assumption that can be easily modified). It is not illogical to admit that the expenses will be paid during the course of the treaty (think of the accounting and claims management of the treaty). So there may be expenses cash flows for all times j.
- variation in the allocated capital ($VC(j)$).
  As announced in the previous section, some capital has to be allocated in order to run the business. However, at the latest at the end of the development, this allocated capital is given back to the shareholders. In practice, the allocation rule may be such that the allocated capital is given back after x years or in function of the evolution of the loss reserves. So there will be variations in the allocated capital, exactly like in the loss reserves.
- return on the allocated capital ($IC(j)$).
  As allocated capital is mobilized, an auto-remuneration of this capital is possible. Indeed the mobilized capital will be invested and will produce a return. Moreover one might think that this auto-remuneration is higher than the remuneration on the loss reserves because the latter are probably invested in risk-free assets. So while capital is allocated there is a cash flow of interest on it at a return rate l.

We are then able to define the cash flows at integer times:

$$CF(j) = \\ CP(j) + B(j) + R(j) + AE(j) + VC(j) + IC(j), \\ j = 0, 1, \dots, n.$$

The problem of tax remains to be treated. In order to find the tax we first have to define the taxable profit at times j and

$j-0.5$:

$$\begin{aligned} TaxProfit(j) &= CP(j)+B(j)+R(j)+AE(j) \\ &\quad +IC(j), \quad j=0,1,\ldots,n, \\ TaxProfit(j-0.5) &= PL(j-0.5)+VR(j-0.5) \\ &\quad +IR(j-0.5), \quad j=1,\ldots,n. \end{aligned}$$

The discounted tax cash flows are then

$$\begin{aligned} Tax(j) &= \tau TaxProfit(j), \quad j=0,\ldots,n, \\ Tax(j-0.5) &= \tau TaxProfit(j-0.5), \quad j=1,\ldots,n. \end{aligned}$$

where τ is an average tax rate. It assumes all cash flows, including financial return, to be taxed at the same rate. This is obviously not always true and specific corrections are easy to include in the model according to the tax regime of the reinsurer's domicile.

The treaty will be acceptable if

$$\sum_{j=0}^{n} \frac{CF(j)-Tax(j)}{(1+c)^j} + \sum_{j=1}^{n} \frac{CF(j-0.5)-Tax(j-0.5)}{(1+c)^{j-0.5}} > 0.$$

6 Allocation of capital

The capital allocated to a line of business is often given as a statutory requirement or as market practice (e.g. capital to premium ratio or capital to reserves ratio). In this section we propose an actuarial way of determining this capital.

Let us assume that the line of business consists in m treaties. For each of these treaties we will adopt the following notation

$$S^{(k)} = S_{X^{\Sigma Re}}(t_{n(k)})_k, \quad k=1,2,\ldots,m.$$

We define the aggregate claims of the line of business

$$S = S^{(1)} + \cdots + S^{(m)}.$$

The technical premium obtained for the line of business is $\mathbb{E}S$. Each piece of the business ($S^{(k)}$) is part of the full business. We have some diversification which reduces the capital need: this is the mutuallisation of insurance.

Let u be the allocated capital. The reinsurer wants to minimize his one year ruin probability, i.e.

$$\mathbb{P}(S > u + \mathbb{E}S) \le \epsilon.$$

More precisely the clients, the employees and the regulatory authorities are interested in the one year ruin probability (see Mayers and Smith (1982) for a detailed explanation on the reasons why corporations (in our case insurance companies) buy insurance contracts (in our case reinsurance contracts)). In particular Mayers and Smith (1982) stress on the fact that well diversified shareholders should not buy insurance contracts. They in fact do because of their clients, employees, ...

Note that one should not try to reduce the one year ruin probability to zero ($\epsilon \to 0$) because it would require too much capital ($u \to \infty$) and the policyholders would not be ready to pay high premiums for such a comfort.

If we know the distributions of $S^{(1)}, S^{(2)}, \ldots, S^{(m)}$ and we assume that the risks are independent (which means that for treaties covering catastrophic risks e.g. another methodology has to be used) then it is not difficult to find the distribution F_S of the aggregate claims by convoluting the $F_{S^{(k)}}$, $k=1,2,\ldots,m$.

Then u is given by

$$u \ge F^{-1}(1-\epsilon) - \mathbb{E}S.$$

Considering the solvency problem, it is enough to know the value of the allocated capital u. However our main concern is to price, so we now need to allocate the capital u treaty by treaty, i.e. we want to find desaggregated capitals $u_1, u_2, \ldots, u_m$ such that $u = \sum_{k=1}^{m} u_k$.

In fact we are just applying the premium calculation from top down (see Bühlmann (1985)).

We want to use the capital as efficiently as possible through a dynamic allocation where each piece of business uses the required risk based capital (u) at any point in time.

Let us choose a premium principle H. We have the following equation

$$\sum_{k=1}^{m} H(S^{(k)}) = u + \mathbb{E}S.$$

This equation can be solved for the unknown parameter of the premium principle. We then obtain the allocated capital per treaty as

$$u_k = H(S^{(k)}) - \mathbb{E}S^{(k)}, \quad k=1,2,\ldots,m.$$

In fact the allocation and release of capital is performed from and to a "capital pool" which is managed at global business level.

It is worth noting that the true one year ruin probability is smaller than ϵ. Indeed, knowing the allocated capital u_k for treaty k, a profit margin will be calculated via the cash flow model in order to remunerate the shareholders. Let us define this profit margin as PM_k , $k=1,2,\ldots,m$. The one year ruin probability then writes

$$\mathbb{P}(S > u + \mathbb{E}S + \sum_{k=1}^{m} PM_k),$$

which is clearly smaller than ϵ.

- If business is good:
$$S \le \mathbb{E}S,$$
then the shareholders are correctly remunerated.
- If the business is bad:
$$\mathbb{E}S \le S \le \mathbb{E}S + \sum_{k=1}^{m} PM_k,$$
then the shareholders are less remunerated than they would have expected.

- If the business is very bad:

$$S > \mathbb{E}S + \sum_{k=1}^{m} PM_k,$$

then the shareholders are not remunerated at all and furthermore a portion of their capital is used to pay the loss.

Regarding the capital allocation there is another important point: capital is not allocated for written premiums. Rather it is allocated to face adverse deviations of the claims. So it seems logical that the desallocation of capital takes some time, e.g. by being proportional to the paid losses.
Note that this presentation does not take into account the financial risk. This is beyond the scope of the present paper and is left to future research.

Note that another option would be to work on the basis of the well known ruin probabilities on a longer horizon. Indeed the classical ruin models (in discrete time) are based on the process:

$$U(t) = u + \pi t - \sum_{i=1}^{t} S_t, \quad t = 1, 2, \ldots$$

i.e. $U(t)$ represents the capital at time t. π is a loaded premium and S_t is assumed to be independent and identically distributed for each t. Implicit hypotheses are that, if business is good, $U(t)$ grows and if business is bad, $U(t)$ decreases. In both cases, there is an economic problem. Indeed the capital is in a closed system, that is when business is good the capital grows too much while when business is very bad it decreases and policyholders would leave the company before it reaches ruin. At the limit, we would apply the methodology of Bühlmann (1985) where he obtains the capital working with the adjustment coefficient. In this case Bühlmann (1985) works on the basis of an upper bound (the Lundberg bound) on the infinite time ruin probability. After a nice mathematical reasoning he obtains a premium which is in fact the pure premium plus a profit margin for the shareholders plus a security loading. It is really not clear what is the economic meaning of this security loading. The reason of the presence of this security loading is obviously the fact that one works with the upper bound on the infinite time ruin probability. In this case, it is indeed well known that there must be a positive security loading in order that the infinite time ruin probability be less than 1.
We therefore definitely suggest to work with the ruin probability after one year.

Note that a better risk measure is the coherent risk measure given by Artzner et al (1999), i.e. the tail conditional expectation:

$$TCE = \mathbb{E}(S|S > q)$$

where q is a certain quantile of the distribution of S. The tail conditional expectation is a coherent measure as described by Artzner et al. (1999). Intuitively it is more interesting than the simple quantile as it gives information on the size of the potential high loss. It explains "how bad is bad". Note that this is in line with the actuarial idea that stop-loss order is more informative than stochastic order. Clearly the capital to allocate can be determined as the tail conditional expectation of the aggregate loss of the company with a predefined quantile q.

It is also worth noting that the capital allocation described in this section may not be coherent. Indeed nothing ensures that coalitions, e.g. $\mathcal{S}$, within the portfolio may be such that

$$\sum_{i \in \mathcal{S}} u_i < TCE(\sum_{i \in \mathcal{S}} S^{(i)}).$$

In cases where the preceding inequality would not be verified, some subportfolios would be able to work more efficiently (in term of capital allocation) without the rest of the portfolio. If subportfolios are profit centers within the company, one may easily understand that the subportfolio's managers would not accept to subsidize some parts of the whole portfolio. We refer to Denault (2001) for further discussions on the coherent allocation of capital.

7 The problem of loading

The present section aims at comparing briefly the premium principles method with the cash flow method.

The premium principle method relies on a function, $H(S)$ which depends on a parameter ξ (e.g. $\xi = \alpha$: expected value premium principle ; $\xi = \gamma$: standard deviation premium principle ; $\xi = \rho$: PH transform premium principle). $H(S)$, the loaded premium, may be rewritten as the pure premium plus a loading. This loading represents the remuneration of allocated capital, expenses, brokerage, retrocession, income on loss reserves, ... The determination of the parameter governing the premium principle is not clear at all, as well as, in fact, the choice of the premium principle. Obviously one would choose the best premium principle with regard to the good properties derived e.g. in Gerber (1979). Nevertheless the choice is not clear. Moreover this approach looks at each piece of business as isolated and does not provide for diversification within the portfolio of the (re)insurer, although this is the essence of the (re)insurance business. Also a volume of capital is required for each piece of business which is obviously not optimal in terms of capital allocation. The latter criticism can be corrected by adapting the premium calculation from top down (Bühlmann (1985)) recalled in the previous section. Indeed let u be the allocated capital and let us assume that the cost of capital is coc. With the notations of the previous section, the allocated capital per treaty is

$$u_k = \frac{H(S^{(k)}) - \mathbb{E}S^{(k)}}{1 + coc},$$

and the premium writes

$$\mathbb{E}S^{(k)} + coc \times u_k,$$

i.e. $coc \times u_k$ represents the profit margin.
The disadvantage of this methodology is that it implicitely assumes that the capital is allocated only during one year, which is not logical. Also all other elements of the pricing are not yet covered: expenses, retrocession, financial revenue, brokerage, ...

On the other hand the financial approach finds the premium such that

$$\sum_{j=0}^{n} \frac{CF(t_j)}{(1+coc)^{t_j}} = 0,$$

where the $CF(t_j)$ cover all cash flows related to the appropriate business. This is a very practical, down to earth approach. It includes all (re)insurance related activities in a comprehensive, explicit and consistent way. It is based on the most general and most widely accepted investment valuation approach and it calls on practical experience of professionnals and of markets to estimate the needed parameter (i.e. the cost of capital). This model allows for realistic simplifications completed with sensitivity analyses to assess the quality of our estimates.

Note that there seems to be a way to introduce some cash flows in the premium principle methodology. Indeed let us load the discounted technical loss:

$$DTL = \sum_{j=1}^{n} \frac{(paid(j-0.5))}{(1+r)^{j-0.5}}.$$

The problem is more complicated if we want to apply a premium principle on the discounted technical loss. Indeed this random variable is a function of n highly correlated random variables $(S_{X^{Re}}(0.5), S_{X^{Re}}(1.5), \ldots, S_{X^{Re}}(n-0.5))$ for which we only calculated the marginal distributions. However it is possible to find the multivariate distribution of the random vector $(S_{X^{Re}}(0.5), S_{X^{Re}}(1.5), \ldots, S_{X^{Re}}(n-0.5))$. Indeed it writes

$$\begin{aligned} S_{X^{\Sigma Re}}(0.5) &= \min(Aal, \\ &\qquad \max(0, \sum_{i=0}^{N} X_i^{\Sigma Re}(0.5) - Aad)) \\ S_{X^{\Sigma Re}}(1.5) &= \min(Aal, \\ &\qquad \max(0, \sum_{i=0}^{N} X_i^{\Sigma Re}(1.5) - Aad)) \\ &\vdots \\ S_{X^{\Sigma Re}}(n-0.5) &= \min(Aal, \\ &\qquad \max(0, \sum_{i=0}^{N} X_i^{\Sigma Re}(n-0.5) - Aad)) \end{aligned}$$

It is not difficult to find the multivariate distribution of the random vector $(X^{\Sigma Re}(0.5), X^{\Sigma Re}(1.5), \ldots, X^{\Sigma Re}(n-0.5))$ because its components are just a function of the same random variable X. Moreover there exists a multivariate Panjer's algorithm giving the distribution of such multivariate compound distributions (see Sundt (1999) or Walhin and Paris (2000)).
Unfortunately the cost for applying the multivariate Panjer's algorithm is extremely high in term of computing time and it needs a multivariate rearithmetization of the random vector $(X(0.5), X(1.5), \ldots, X(n-0.5))$ in order to be able to work efficiently. So we obtain with enormous difficulties a really small enhancement of the model.

We definitely would recommend to work within the cash flow setting and to use premium principles approach in order to allocate capital.

8 Clauses making the reinsurance premium random

Often one observes in excess of loss treaties that the reinsurance premium is a function of the excess of loss amounts. In these situations, governed by typical clauses, the reinsurance premium is a random variable. We will describe three cases of such clauses. We first describe the clauses from a theoretical point of view and then we show how to adapt the cash flow model in order to take these clauses into account in the practical pricing.

In this section the reinsurance premium consists in a fixed premium P^{Init} and a random premium: P^{Rand}.

$$P = P^{Init} + P^{Rand}.$$

8.1 Paid reinstatements

Often the annual aggregate limit clause takes another form. The annual aggregate limit (Aal) is a multiple of the length of the layer L. Assume that this multiple is $v+1$. It means that the layer may be consumed $v+1$ times. We say that there are v reinstatements. These reinstatements may be free or paid. The case of free reinstatements is just a pure annual aggregate limit. In case of paid reinstatements, the most common situation is that reinstatement premiums are calculated as a fraction ρ_u , $1 \leq u \leq v$ of the initial reinsurance premium P^{Init} and in proportion of the part of the layer hit by the claims. We say that the reinstatement premiums are payable pro rata of the amount consumed (pro rate capita).
The payment of the reinstatement premium is compulsory.
This type of clause is very common in property reinsurance.
The reinstatement premiums are given by

$$P^{Rand} = \frac{P^{Init}}{L} \sum_{u=1}^{v} \rho_u \min(L, \max(0, S-(u-1)L)).$$

They represent the random part of the reinsurance premium and are therefore denoted P^{Rand}. The retained risk of the reinsurer is

$$S^{Clau} = \min(S, (v+1)L) - P^{Rand}.$$

Then the initial part of the premium is given by

$$P^{Init} = H(S^{Clau}).$$

The case of the pure premium has been discussed by Sundt (1991) whereas the case of the PH-transform premium principle has been discussed in Walhin and Paris (2001). The case of the standard-deviation premium principle is discussed in both papers. If $H(S) = (1+\alpha)\mathbb{E}S$, then it is possible to find P^{Init} analytically. Otherwise an iterative technique has to be built (see Walhin and Paris (2001)).

8.2 Sliding scale premium

For any type of reinsurance, the ceding company may ask for a sliding scale premium if it believes that the aggregate claims will not be important. It is also useful when the loss distribution is difficult to estimate, in which case a fraction of the variability is returned to the cedant.
The reinsurance premium writes:

$$P = \begin{cases} P_{min} & \text{if} \quad S \leq \frac{P_{min}}{f}, \\ fS & \text{if} \quad \frac{P_{min}}{f} < S < \frac{P_{max}}{f}, \\ P_{max} & \text{if} \quad S \geq \frac{P_{max}}{f}, \end{cases}$$

where f is a loading coefficient (indeed if the aggregate loss is within the limits of the scale, the reinsurer will receive as a premium the loss plus a loading for its expenses, cost of capital, ...). Generally $f = \frac{100}{70}$ or $\frac{100}{80}$.

$$P^{Rand} = \begin{cases} 0 & \text{if} \quad S \leq \frac{P_{min}}{f}, \\ fS - P_{min} & \text{if} \quad \frac{P_{min}}{f} < S < \frac{P_{max}}{f}, \\ P_{max} - P_{min} & \text{if} \quad S \geq \frac{P_{max}}{f}. \end{cases}$$

Note that this kind of clause may be applied to any type of reinsurance, not only to the excess of loss reinsurance. The aggregate claims distributions and the premiums are given by

$$\begin{aligned} S^{Clau} &= S - P^{Rand}, \\ P^{Init} &= H(S^{Clau}). \end{aligned} \tag{8}$$

In practice, P_{min} is given and P_{max} has to be found by a numerical technique whatever the chosen premium principle, i.e. solving (8) for P^{Init}. If $H(S) = \mathbb{E}S$ then P_{max} is evaluated in such a way that the average premium with the sliding scale equals the premium in the classical fixed premium treaty.

8.3 Profit commission

Sometimes the reinsurer offers a profit participation although the professional market does not like this situation for excess of loss covers. This means that after the aggregate claims have been observed, in case of a good statistic, the reinsurer gives back a share of the reinsurance premium to the ceding company. This kind of clause is not typical of excess of loss reinsurance.
A common type of profit commission is to give back

$$PC = \alpha \max(0, \beta P^{Init} - S),$$

where $1-\beta$ represents the administrative costs of the reinsurer and α represents the fraction of the profit of the reinsurer that is given back to the ceding company. The aggregate part and initial premium of the reinsurer become:

$$\begin{aligned} S^{Clau} &= S + PC, \\ P^{Init} &= H(S^{Clau}). \end{aligned} \tag{9}$$

The premium has to be found by an iterative technique, whatever the chosen premium principle, i.e. solving (9) for P^{Init}. If $H(S) = \mathbb{E}S$, the equation to solve is such that the expected premium when there is a profit participation is the same than the fixed premium.
Obviously there exists other types of profit participations but we limit ourselves to the present type in this paper.

8.4 The practical pricing

The formulae given hereabove are theoretical. With these formulae we are able to find the technical premium of a treaty with such clauses by applying the formulae to $S = S_{X^{\Sigma Re}}(n)$ with $H(S) = \mathbb{E}S$.
However the premium with such a random clause has to take into account all the technical, financial, economic and commercial elements of the pricing. In particular, with such clauses, there will be premium adjustments in the future. This implies new cash flows regarding the premiums.
We will proceed in two steps. The first one is really simple: we just calculate the commercial premium necessary to cover the treaty if there is no "random" clause. We then obtain the evolution of paid losses, loss reserves, interests on loss reserves, allocated capital, interests on allocated capital and administrative expenses. There are no reasons to believe that these elements will be different in the cash flow model with "random clause". So we move to the second step, i.e. the cash flow model with the "random" clause. The previous elements are fixed. Other elements may vary: premiums, brokerage, retrocession, and taxes. The process will be iterative. As a first guess we choose an initial premium (or one limit of the scale in the case of a sliding scale). According to the evolution of the incurred losses, this premium will be split in several premiums in the future, i.e.

- $CP(0) = P^{Init}$ (or, more exactly the minimum and deposit premium, the balance of which being paid in $t = 1$) for a treaty with paid reinstatements. $CP(j) =$ future adjustments for reinstatements due to incurred losses hitting the layer for $j = 1, 2, \ldots, n$.
- $CP(0) = P^{Init} = P_{min}$ (or, more exactly the minimum and deposit premium, the balance of which being paid in $t = 1$) for a treaty with sliding scale. $CP(j) =$ future adjustments for $j = m, m+1, \ldots, n$ where m is the first year for which a premium adjustment is contractually agreed.

- $CP(0) = P^{Init}$ (or, more exactly the minimum and deposit premium, the balance of which being paid in $t = 1$) for a treaty with profit commission. $CP(j)$ = future adjustments for profit commission for $j = m, m+1, \ldots, n$ where m is the first year for which a premium adjustment is contractually agreed.

With this pattern of premium payments, we immediately obtain the pattern of brokerage, retrocession and as a result the pattern of tax. We are then able to calculate the net present value of the business. If it is positive we try a new premium lower than the previous one. If it is negative we try a new premium higher than the previous one. The trial and error scheme is continued until the net present value of the business is 0.

9 Numerical application

In this section we will present a numerical application for which we will calculate the reinsurance premium. We will also proceed to some sensitivity analyses and alternative pricings in section 10. The elements for the pricing are the following:

- the distribution of the claim amounts, X, is Pareto with parameters $A = 400$ and $\alpha = 1.50$:

$$\begin{aligned} F_X(x) &= 0 \quad \text{if } x \le A, \\ &= 1 - \left(\frac{x}{A}\right)^{-\alpha} \quad \text{if } x > A. \end{aligned}$$

  The Pareto distribution is commonly used by excess of loss reinsurers to modelize large losses.
- the distribution of the claim numbers, N is Poisson with parameter $\lambda = 2.5$:

$$\mathbb{P}[N = n] = p(n) = e^{-\lambda}\frac{\lambda^n}{n!}, \quad n = 0, 1, \ldots$$

  The Poisson distribution is also often used by reinsurers. Indeed the occurrence of large losses may be thought of as purely random which the Poisson distribution modelizes well. Moreover we noticed in section 2 that the calculations are easy if the counting distribution belongs to Panjer's class, which is the case for the Poisson distribution.
- the lattice span is taken as $h = 25$.
- the claims are completely developed in $n = 7$ years with the following claims payment pattern:

t	0	1	2	3	4	5	6	7
c	5%	10%	10%	10%	25%	25%	10%	5%
c^Σ	5%	15%	25%	35%	60%	85%	95%	100%

- the overstatement pattern of the ceding company is:

t	0	1	2	3
d	125%	125%	125%	125%

t	4	5	6	7
d	105%	105%	100%	100%

- the future inflation is modelled by a geometric growth of 3% a year, i.e. $\frac{inf(j)}{inf(j-1)} - 1 = 3\%$, $j = 1, 2, \ldots, n$. We furthermore assume that $inf(j - 0.5) = inf(j)$, $j = 1, 2, \ldots, n$.
- the future inflation and superimposed inflation is modelled by a geometric growth of 4.5% a year, i.e. $\frac{supinf(j)}{supinf(j-1)} - 1 = 4.5\%$, $j = 1, 2, \ldots, n$. We furthermore assume that $supinf(j - 0.5) = supinf(j)$, $j = 1, 2, \ldots, n$.
- the interest rate obtained on the loss reserve is supposed to be $r = 5\%$.
- the return obtained on the allocated capital is supposed to be $l = 7\%$.
- the cost of capital is assumed to be $coc = 11\%$.
- the allocated capital, $C(j)$, $j = 0, 1, \ldots, n$ is assumed to be 1.25 times the standard deviation of the ultimate aggregate claims, i.e. $\sqrt{\mathbb{V}ar(1-\nu)S_{X^{\Sigma Re}}(n - 0.5)}$ where ν is the fraction of the claims paid by the retrocessionnaire. We make the hypothesis that capital has to be allocated during three years.
- the priority of the treaty is $P = 500$.
- the limit of the treaty is $L = 2500$.
- the date of payment stability clause is applied on the priority and on the limit with a margin of 10%. It is also assumed that the application of the stability clause is based on incurred losses.
- there is no annual aggregate deductible and no annual aggregate limit.
- the interests sharing clause is used and we assume that the portion of interests in the losses is $\delta = 15\%$.
- the estimated premium income is assumed to be 50000. This information is important as the reinsurance premium is usually expressed as a percentage of the cedant's premium income. One classicaly speaks of a rate.
- the share of the reinsurer in the treaty is assumed to be 20%. It is indeed common practice that several reinsurers take a share in a given treaty. Unless the ceded risk is really small, a cedant would not accept to work with only one reinsurer for solvency reasons.
- brokerage is 10%.
- there is a minimum and deposit premium of 80% of the expected commercial reinsurance premium. By deposit we mean that 80% of the premium is paid at time $t = 0$ whereas the balance is paid at time $t = 1$. By minimum we mean that at least the reinsurance rate times 80% of the premium income (estimated by the cedant) will be paid. In case the actual premium income is lower than 80% of the estimated premium income, the minimum and deposit premium is due. We assume that the estimated premium income will be the actual one.
- retrocession costs are 3% of the commercial premium. We assume that on average $\nu = 2\%$ of the claims are paid by the retrocession.
- administrative expenses are 5 for the fixed part and 4% of the paid losses each year (the proportional administrative expenses are assumed to be paid at the end of the year).
- the average tax rate is assumed to be 30%.

The following table gives the evolution of the from ground up (fgu) cumulative payments for the smallest possible losses given by the discretization ($Y(j)$ denotes $X^\Sigma(j)$, see (1)):

X	$Y(0.5)$	$Y(1.5)$	$Y(2.5)$	$Y(3.5)$
400	20	61.80	105.48	151.13
425	21.25	65.66	112.07	160.57
450	22.5	69.52	118.67	170.02
475	23.75	73.39	125.26	179.46
500	25	77.25	131.85	188.91
525	26.25	81.11	138.44	198.36
550	27.50	84.98	145.04	207.80

X	$Y(4.5)$	$Y(5.5)$	$Y(6.5)$	$Y(7.5)$
400	270.38	395.00	447.09	474.31
425	287.28	419.69	475.03	503.95
450	304.18	444.37	502.97	533.59
475	321.08	469.06	530.92	563.24
500	337.97	493.75	558.86	592.88
525	354.87	518.43	586.80	622.53
550	371.77	543.12	614.75	652.17

The following table gives the evolution of the loss reserves, including the overstatement pattern ($Y(j)$ denotes $d(j)R(j)$):

X	$Y(0.5)$	$Y(1.5)$	$Y(2.5)$	$Y(3.5)$
400	567.88	513.28	461.03	403.97
420	603.37	545.36	489.84	429.22
450	638.87	577.44	518.66	454.47
475	674.36	609.52	547.47	479.72
500	709.85	641.60	576.29	504.97
525	745.34	673.68	605.10	530.21
550	780.84	705.76	633.92	555.46

X	$Y(4.5)$	$Y(5.5)$	$Y(6.5)$	$Y(7.5)$
400	214.12	83.27	27.22	0
420	227.50	88.48	28.92	0
450	240.89	93.68	30.62	0
475	254.27	98.89	32.32	0
500	267.65	104.09	34.02	0
525	281.04	109.30	35.72	0
550	294.42	114.50	37.42	0

The next table gives the evolution of the incurred loss from ground up ($Y(j)$ denotes $I(j)$, see (2)):

X	$Y(0.5)$	$Y(1.5)$	$Y(2.5)$	$Y(3.5)$
400	587.88	576.96	566.51	555.10
420	624.62	613.02	601.92	589.79
450	661.37	649.08	637.33	624.49
475	698.11	685.14	672.73	659.18
500	734.85	721.20	708.14	693.87
525	771.59	757.26	743.55	728.57
550	808.34	799.32	778.95	763.26

X	$Y(4.5)$	$Y(5.5)$	$Y(6.5)$	$Y(7.5)$
400	484.50	478.27	474.31	474.31
420	514.78	508.16	503.95	503.95
450	545.06	538.05	533.59	533.59
475	575.35	567.95	563.24	563.24
500	605.63	597.84	592.88	592.88
525	635.91	627.73	622.53	622.53
550	666.19	657.62	652.17	652.17

The following table gives the evolution of the deductible and limit due to the stability clause (see (3) and (4)):

P/L	0.5	1.5	2.5	3.5
500	500	500	500	500
2500	2500	2500	2500	2500

P/L	4.5	5.5	6.5	7.5
500	541.55	547.96	550.24	551.09
2500	2707.77	2739.80	2751.21	2755.45

Without interests sharing clause, the evolution of reinsurance payments would be

X	$Y(0.5)$	$Y(1.5)$	$Y(2.5)$	$Y(3.5)$
400	0	0	0	0
425	0	0	0	0
450	0	0	0	0
475	0	0	0	0
500	0	0	0	0
525	0	0	0	0
550	0	0	0	0

X	$Y(4.5)$	$Y(5.5)$	$Y(6.5)$	$Y(7.5)$
400	0	0	0	0
425	0	0	0	0
450	0	0	0	0
475	0	0	0	12.15
500	0	0	0	41.79
525	0	0	36.56	71.43
550	0	0	64.50	101.08

Without interests sharing clause, the evolution of reinsurance incurred losses would be

X	$Y(0.5)$	$Y(1.5)$	$Y(2.5)$	$Y(3.5)$
400	87.88	76.96	66.51	55.10
425	124.62	113.02	101.92	89.79
450	161.37	149.08	137.33	124.49
475	198.11	185.14	172.73	159.18
500	234.85	221.20	208.14	193.87
525	271.59	257.26	243.55	228.57
550	308.34	293.32	278.95	263.26

X	$Y(4.5)$	$Y(5.5)$	$Y(6.5)$	$Y(7.5)$
400	0	0	0	0
425	0	0	0	0
450	3.51	0	0	0
475	33.79	19.99	13.00	12.15
500	64.07	49.88	42.64	41.79
525	94.36	79.77	72.28	71.43
550	124.64	109.66	101.93	101.08

The following table gives the liability of the reinsurer in the principal of the losses (legal interests excluded) with respect to the payments ($Y(j)$ denotes $\min(L(j), \max(0, (1-\delta)X^{\Sigma}(j) - P(j))))$):

X	$Y(0.5)$	$Y(1.5)$	$Y(2.5)$	$Y(3.5)$
400	0	0	0	0
425	0	0	0	0
450	0	0	0	0
475	0	0	0	0
500	0	0	0	0
525	0	0	0	0
550	0	0	0	0

X	$Y(4.5)$	$Y(5.5)$	$Y(6.5)$	$Y(7.5)$
400	0	0	0	0
425	0	0	0	0
450	0	0	0	0
475	0	0	0	0
500	0	0	0	0
525	0	0	0	0
550	0	0	0	3.25

We observe that an effect of the interest sharing clause is to let disappear the smallest claims from the reinsurer's liability.
The following table gives liability of the reinsurer in the principal of the losses (legal interests excluded) with respect to the

	0	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
TFP	**196.50**	0	0	0	0	0	0	0	0
PL	0	-2.08	-9.73	-13.36	-18.00	-56.07	-76.01	-33.52	-19.02
VR	0	-328.51	19.44	20.54	28.21	121.82	81.86	37.44	19.21
IR	0	0	16.43	15.45	14.43	13.02	6.93	2.83	0.96
$CF(j)$	196.5								
$CF(j+0.5)$	0	-330.59	26.13	22.63	24.64	78.77	12.78	6.75	1.15
$\frac{CF(j)}{(1+coc)^j}$	196.5	0	0	0	0	0	0	0	0
$\frac{CF(j+0.5)}{(1+coc)^{j+0.5}}$	0	-313.78	22.35	17.44	17.10	49.25	7.20	3.43	0.53
NPV	**0**								

Table 1. Cash flow model for the technico-financial premium

incurred loss (Y is given by (5)):

X	$Y(0.5)$	$Y(1.5)$	$Y(2.5)$	$Y(3.5)$
400	0	0	0	0
420	30.93	21.07	11.63	1.32
450	62.16	51.72	41.73	30.81
475	93.39	82.37	71.82	60.30
500	124.62	113.02	101.92	89.79
525	155.86	143.67	132.01	119.28
550	187.09	174.32	162.11	148.77

X	$Y(4.5)$	$Y(5.5)$	$Y(6.5)$	$Y(7.5)$
400	0	0	0	0
420	0	0	0	0
450	0	0	0	0
475	0	0	0	0
500	0	0	0	0
525	0	0	0	0
550	24.71	11.02	4.10	3.25

The next table gives the total liability (including legal interests after the interest sharing clause) of the reinsurer in the payments (Y is given by (7)):

X	$Y(0.5)$	$Y(1.5)$	$Y(2.5)$	$Y(3.5)$
400	0	0	0	0
425	0	0	0	0
450	0	0	0	0
475	0	0	0	0
500	0	0	0	0
525	0	0	0	0
550	0	0	0	0

X	$Y(4.5)$	$Y(5.5)$	$Y(6.5)$	$Y(7.5)$
400	0	0	0	0
425	0	0	0	0
450	0	0	0	0
475	0	0	0	0
500	0	0	0	0
525	0	0	0	0
550	0	0	0	3.83

The next table gives the total liability (including legal interests after the interest sharing clause) of the reinsurer in the incurred losses (Y is given by (6)):

X	$Y(0.5)$	$Y(1.5)$	$Y(2.5)$	$Y(3.5)$
400	0	0	0	0
420	36.39	24.79	13.68	1.56
450	73.13	60.85	49.09	36.25
475	109.87	96.91	84.50	70.95
500	146.62	132.97	119.90	105.64
525	183.36	169.03	155.31	140.33
550	220.10	205.09	190.72	175.03

X	$Y(4.5)$	$Y(5.5)$	$Y(6.5)$	$Y(7.5)$
400	0	0	0	0
420	0	0	0	0
450	0	0	0	0
475	0	0	0	0
500	0	0	0	0
525	0	0	0	0
550	29.07	12.96	4.83	3.83

The following table gives the expected aggregate payments and loss reserves of the reinsurer.

	0.5	1.5	2.5	3.5
$-PL$	10.38	48.65	66.80	89.98
RES	1642.56	1545.39	1442.70	1301.65

	4.5	5.5	6.5	7.5
$-PL$	280.35	380.03	167.61	95.10
RES	692.54	283.24	96.05	0

By adding up the payments we immediately arrive at the technical rate (TR):

$$TR = \frac{1138.90}{50000} = 2.28\%.$$

As discussed earlier, this rate is not satisfactory because it does not take into account the interests the reinsurer can obtain on loss reserves. On the other hand it also does not take into account the cost of reserving (in particular when there is overstatement). Finally it does not take into account the fact that the total payment is a sum of different cash flows. This is the reason why we introduce the technico-financial premium. Table 1 gives the cash flow model giving the technico-financial premium. This table takes into account a reinsurers's share of 20%.

The technico-financial rate is thus given by

$$TFR = \frac{196.50}{50000 \times 20\%} = 1.97\%.$$

We now obtain the commercial premium as shown in Table 2. The total commercial premium is then

$$258.80 + 64.70 = 323.50,$$

which produces a rate of

$$\frac{323.50}{50000 \times 20\%} = 3.24\%.$$

Summarizing we have the following rates:

TR	2.28%
TFR	1.97%
CR	3.24%

Note that the total commercial premium is the variable we use in order to find a $NPV = 0$ iteratively. $CP(0)$ and $CP(1)$ are linked to the total commercial premium by

$$CP(0) = 80\% \times 323.50$$
$$CP(1) = 20\% \times 323.50$$

j	0	1	2	3	4	5	6	7	8
$j-0.5$	0	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
CP	258.80	64.70	0	0	0	0	0	0	0
AE	-5	-0.08	-0.39	-0.53	-0.72	-2.24	-3.04	-1.34	-0.76
B	-25.88	-6.47	0	0	0	0	0	0	0
R	-7.76	-1.90	0.19	0.27	0.36	1.12	1.52	0.67	0.38
PL	0	-2.08	-9.73	-13.36	-18.00	-56.07	-76.01	-33.52	-19.02
VR	0	-328.51	19.44	20.54	28.21	121.82	81.86	37.44	19.21
IR	0	0	16.43	15.45	14.43	13.02	6.93	2.83	0.96
VC	-337.01	0	0	337.01	0	0	0	0	0
IC	0	23.59	23.59	23.59	0	0	0	0	0
$CF(j)$	-116.86	79.84	23.40	360.33	-0.36	-1.12	-1.52	-0.67	-0.38
$CF(j+0.5)$	0	-330.59	26.13	22.63	24.64	78.77	12.78	6.75	1.15
$TaxPr(j)$	220.16	79.84	23.40	23.32	-0.36	-1.12	-1.52	-0.67	-0.38
$TaxPr(j+0.5)$	0	-330.59	26.13	22.63	24.64	78.77	12.78	6.75	1.15
$Tax(j)$	66.05	23.95	7.02	7.00	-0.11	-0.34	-0.46	-0.20	-0.11
$Tax(j+0.5)$	0	-99.18	7.84	6.79	7.39	23.63	3.83	2.02	0.34
$\frac{CF(j)-Tax(j)}{(1+coc)^j}$	-182.90	50.35	13.29	258.36	-0.17	-0.47	-0.57	-0.23	-0.12
$\frac{CF(j+0.5)-Tax(j+0.5)}{(1+coc)^{j+0.5}}$	0	-219.65	15.64	12.20	11.97	34.47	5.04	2.40	0.37
NPV	**0**								

Table 2. Cash flow model for the commercial premium

This is merely the application of a minimum and deposit premium of 80%.

10 Sensitivity analyses

Now let us change some pricing conditions in order to analyse the differences in the pricing. All comparisons are assumed to be made with the numerical example of the previous section unless otherwise stated.

- Let us assume a new overstatement pattern:

t	0	1	2	3
d	150%	150%	125%	125%
t	4	5	6	7
d	105%	105%	100%	100%

We obtain:

TR	TFR	CR
2.28%	2.05%	3.34%

- Let us assume no overstatement pattern.
We obtain:

TR	TFR	CR
2.28%	1.82%	3.07%

Clearly overstating has a cost in term of TFR and CR. This is what we observe on the above-mentioned pricings.

- Let us assume a faster payment pattern:

t	0	1	2	3	4	5	6	7
c	30%	25%	20%	15%	5%	5%	0%	0%

We obtain:

TR	TFR	CR
2.05%	1.84%	3.02%

- Let us assume a slower payment pattern:

t	0	1	2	3	4	5	6	7
c	0%	0%	5%	5%	15%	20%	25%	30%

We obtain:

TR	TFR	CR
2.47%	2.08%	3.42%

It may be tempting to believe that the slower the payment pattern, the smaller the TFR and CR. This example shows it is not necessarily the case. Indeed one should not forget that

- there is a negative effect of inflation if the payment pattern is slow, which is indicated in the value of the TR. Moreover, higher TR may imply higher allocated capital with our assumption that allocated capital is a function of the standard deviation of the ultimate loss.
- overstatement has a negative effect on the rates.

- Let us assume that there is no overstatement and let us vary the speed of payments:

	TR	TFR	CR
Slow	2.47%	1.89%	3.20%
Normal	2.28%	1.82%	3.07%
Fast	2.05%	1.78%	2.95%

We note that even without overstatement the CR decreases with the speed of payments, in our numerical example.

- Let us assume that there is no overstatement and that there is no superimposed inflation :

	TR	TFR	CR
Slow	2.19%	1.68%	2.90%
Normal	2.10%	1.68%	2.87%
Fast	1.97%	1.72%	2.87%

We observe that with a lower inflation of the claims, the effect on CR and TFR is reduced. Even CR seems to decrease with the speed of payments. However let us observe that the TFR does not decrease. So it may seem curious that the CR decreases. The reason is to be found in the capital allocation:

	C
Slow	340.64
Normal	324.65
Fast	301.96

Clearly a lower allocated capital implies less profit to the shareholders and this explains why even if the TFR increases, the CR may decrease.

- Let us assume that the part of the interest in the losses is now $\delta = 25\%$ We obtain:

TR	TFR	CR
2.05%	1.77%	2.98%

- Let us assume that there is no interest sharing clause. We obtain:

TR	TFR	CR
2.60%	2.23%	3.59%

We observe here the dramatic effect of the interests sharing clause.

- Let us assume that there is no stability clause. We obtain:

TR	TFR	CR
2.40%	1.97%	3.30%

Rates are higher, as expected.

- Let us assume that there is a stability clause without margin. We obtain:

TR	TFR	CR
2.26%	1.94%	3.21%

Rates are lower, as expected.

- Let us assume that there is a stability clause with a SIC of 10%. We obtain:

TR	TFR	CR
2.36%	2.01%	3.28%

Rates are now higher, as expected. Note that due to the relatively low level of (super)imposed inflation and the low number of development years, the SIC has almost the same effect than no index clause.

- Let us assume that the stability clause is applied on basis of paid losses only. We obtain :

TR	TFR	CR
2.28%	1.97%	3.24%

Rates are similar which is not illogical. Indeed the final deductible and limits are the same for both models. Only minor differences exist in the life of the treaty due to the overstatement, implying some cash flows paid at different moments. Nevertheless the influence is, in our case, negligible.

- Let us assume that the stability clause is applied only on the priority. We obtain :

TR	TFR	CR
2.22%	1.93%	3.15%

Rates are lower, as expected. Indeed, if the limit is not indexed, the cover is smaller for the ceding company, implying a smaller rate.

- Let us assume that the allocated capital is released after 5 years We obtain:

TR	TFR	CR
2.28%	1.97%	3.67%

- Let us assume that the allocated capital is released after 7 year We obtain:

TR	TFR	CR
2.28%	1.97%	4.02%

As expected when capital has to be allocated on a longer period, the profit margin for the shareholders is higher which makes the CR higher.

- Let us assume that the allocated capital is given by 2 times the standard deviation of the ultimate aggregate claims. We obtain:

TR	TFR	CR
2.28%	1.97%	3.74%

As expected, a higher allocated capital implies a higher profit margin for the shareholders.

- Let us assume that no allocated capital is needed. We find

TR	TFR	CR
2.28%	1.97%	2.39%

The difference between this premium and the one we found is just the profit the shareholders demand to let the business run. Indeed if one keeps the premium calculated with allocated capital, the cash flow model brings a $NPV = 50.24$. This is exactly the profit margin of the shareholders. It can also be found by:

$$\frac{C(coc - l(1-\tau))}{1+coc} + \frac{C(coc - l(1-\tau))}{(1+coc)^2} + \frac{C(coc - l(1-\tau))}{(1+coc)^3},$$

i.e. the discounted returns demanded by the shareholders as long as capital is allocated.

- Let us assume that the minimum and deposit premium is now 60%. We obtain:

TR	TFR	CR
2.28%	1.97%	3.30%

The commercial rate is now higher, which is logical because a larger fraction of the premium will be paid at time $t = 1$ which has a cost in term of discounting.

- Let us assume that the minimum and deposit premium is 100%. We obtain:

TR	TFR	CR
2.28%	1.97%	3.17%

- Let us assume that there is no tax. We obtain:

TR	TFR	CR
2.28%	1.97%	2.78%

- Let us assume that the share of the reinsurer is now 40%. We obtain:

TR	TFR	CR
2.28%	1.97%	3.21%

The fixed costs are now less important (in proportion of the premium) and so the CR decreases.

- Let us assume that the estimated premium income is 100000 and that, consequently, λ is doubled, i.e. $\lambda = 5$. We obtain:

TR	TFR	CR
2.28%	1.97%	2.96%

j	0	1	2	3	4	5	6	7	8
$j-0.5$	0	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
CP	160.00	40.00	0	0	216.05	−31.05	−3.47	−2.05	0
AE	−5	−0.08	−0.39	−0.53	−0.72	−2.24	−3.04	−1.34	−0.76
B	−16.00	−4.00	0	0	−21.16	3.10	0.35	0.21	0.00
R	−4.80	−1.16	0.19	0.27	−6.12	2.05	1.62	0.73	0.38
PL	0	−2.08	−9.73	−13.36	−18.00	−56.07	−76.01	−33.52	−19.02
VR	0	−328.51	19.44	20.54	28.21	121.82	81.86	37.44	19.21
IR	0	0	16.43	15.45	14.43	13.02	6.93	2.83	0.96
VC	−337.01	0	0	337.01	0	0	0	0	0
IC	0	23.59	23.59	23.59	0	0	0	0	0
$CF(j)$	−202.81	58.35	23.40	360.33	187.60	−28.13	−4.54	−2.45	−0.38
$CF(j+0.5)$	0	−330.59	26.13	22.63	24.64	78.77	12.78	6.75	1.15
$TaxPr(j)$	134.20	58.35	23.40	23.32	187.60	−28.13	−4.54	−2.45	−0.38
$TaxPr(j+0.5)$	0	−330.59	26.13	22.63	24.64	78.77	12.78	6.75	1.15
$Tax(j)$	40.26	17.50	7.02	7.00	56.28	−8.44	−1.36	−0.74	−0.11
$Tax(j+0.5)$	0	−99.18	7.84	6.79	7.39	23.63	3.83	2.02	0.34
$\frac{CF(j)-Tax(j)}{(1+coc)^j}$	−243.07	40.84	16.38	353.34	131.32	−19.69	−3.18	−1.72	−0.27
$\frac{CF(j+0.5)-Tax(j+0.5)}{(1+coc)^{j+0.5}}$	0	−231.41	18.29	15.84	17.25	55.14	8.95	4.72	0.80
NPV	**0**								

Table 3. Cash flow model for the commercial premium with a sliding scale

As there is no Aal and Aad, TR and TFR are the same. CR is lower because:

- the allocated capital is 476.61 which is less than twice the allocated capital in the original model.
- the fixed costs are proportionally less important

- Let us assume there is a similar variation of $x\%$ for inflation, inflation + superinflation, interest on loss reserves, interest on allocated capital and cost of capital. We obtain:

	TR	TFR	CR
−2%	2.15%	1.95%	3.10%
−1%	2.22%	1.97%	3.18%
0%	2.28%	1.97%	3.24%
+1%	2.36%	1.96%	3.30%
+2%	2.44%	1.96%	3.37%

Obviously, the TR increases with (superimposed) inflation. In this example we observe that the TFR remains about the same whatever the value of the rates.

Regarding the CR it still increases with the rates, showing the big influence of the cost of capital on the pricing.

- Let us now assume that there is an annual deductible aggregate of $Add = 500$. We obtain:

TR	TFR	CR
1.63%	1.42%	2.51%

As expected, application of an annual aggregate deductible lets the premium decrease. Note that the financial discount, $\frac{1.42\%}{1.63\%} = 86.99\%$ whereas it is 86.27% in the original model. This might seem curious. Indeed one would expect a larger discount factor (DF) in case of an annual aggregate deductible. This is not the case in our numerical example due to the overstatement pattern.

In the case where there is no overstatement pattern we obtain the following discount factors in function of the annual aggregate deductible:

Aad	$1-DF$
0	80.03%
500	79.42%
1000	78.88%
1500	78.27%

For this case we observe that the discount factor is increasing with the annual aggregate deductible.

- Let us now assume that there is an annual aggregate limit of $Aal = 10000$. We obtain:

TR	TFR	CR
2.28%	1.97%	3.23%

We obtain almost the same rates than without the annual aggregate limit. This is because the chance that the aggregate claims is higher than Aal is extremely low.

- Let us assume that the span has changed. We obtain:

h	TR	TFR	CR
100	2.31%	1.97%	3.24%
50	2.28%	1.96%	3.24%
25	2.28%	1.97%	3.24%
10	2.28%	1.97%	3.24%
5	2.29%	1.97%	3.25%

This table shows that the choice $h = 25$ is adequate fot the application of the multiple Panjer's recursions.

We now present the pricing for the case of a sliding scale. We always assume the same conditions. The sliding scale has a minimum rate $R_{min} = 2\%$, a loading $f = \frac{100}{70}$ and we look for the maximum rate R_{max}. We also assume that the first premium adjustement is foreseen after four years. The solution is given by $R_{max} = 4.79\%$. Table 3 gives the cash flow model giving the commercial premium.

We observe the particular pattern of premium payment. At time $t = 0$, 80% of the minimal premium is paid. At time $t = 1$, 20% of the minimal premium is paid. There are no adjustments until time $t = 4$. At that time a huge positive adjustment is needed after what smaller negative adjustments follow. This shows an important fact for the sliding scale: a fraction of the premium may be paid late and this must have an influence on the pricing.

In the next table we give R_{max} in function of R_{min} and the first time for premium adjustments (m):

R_{min}/m	1	2	3	4
1.00%	4.36%	4.84%	5.40%	6.05%
1.50%	4.13%	4.50%	4.92%	5.42%
2.00%	3.89%	4.14%	4.44%	4.79%
2.50%	3.63%	3.78%	3.95%	4.15%
3.00%	3.36%	3.41%	3.46%	3.53%

This table confirms what was said above. We observe a dramatic effect of the variable first year of premium adjustment. This aspect is however classicaly neglected by reinsurers when pricing sliding scale covers.

We now present the commercial premium rate for the treaty with a profit participation in function of α, β and the first year of premium adjustment (m). The first table is for $\beta = 50\%$:

α/m	1	2	3	4
80%	5.61%	5.36%	5.13%	4.92%
60%	5.13%	4.92%	4.73%	4.56%
40%	4.59%	4.43%	4.29%	4.16%
20%	3.96%	3.87%	3.80%	3.73%

The second table is for $\beta = 75\%$:

α/m	1	2	3	4
80%	7.19%	6.78%	6.40%	6.06%
60%	6.40%	6.06%	5.75%	5.46%
40%	5.50%	5.25%	5.02%	4.81%
20%	4.47%	4.32%	4.19%	4.07%

Once again we observe the effect of the variable first year of adjustment premium. In the case of a profit participation, a higher initial premium is needed and future negative adjustments will be provided. This shows that smaller rates can be proposed to the cedant if one agrees on late premium adjustments.
We now present the commercial rate for the treaty with $v = 3$ reinstatements (this corresponds to an $Aal = 10000$, which was seen to be a sufficient cover for the cedant):

$price$	CR
0%; 0%; 0%	3.23%
100%; 100%; 100%	2.56%
200%; 200%; 200%	2.12%
0%; 100%; 100%	3.11%
0%; 0%; 100%	3.22%

The case $0\%; 0\%; 0\%$ is obviously identical to the case $Aal = (3 + 1) \times 2500 = 10000$.
We observe the big influence of the paid reinstatements.
We now present the commercial rate for the treaty with 3 reinstatements in the case where there is no brokerage on the reinstatement premiums:

$price$	CR
0%; 0%; 0%	3.23%
100%; 100%; 100%	2.50%
200%; 200%; 200%	2.03%
0%; 100%; 100%	3.09%
0%; 0%; 100%	3.22%

As expected, the influence of the absence of brokerage on reinstatement premiums is important only for expensive reinstatements.

11 Conclusion

We have shown in this paper that a comprehensive methodology is of great help when pricing excess of loss treaties. Indeed these treaties are often assorted with typical clauses that make them difficult to price.
In particular the long tail business implies that cash flow models should be used. Note that even for short tail business it should be the case.
All the elements of a pricing are combined in a unique tool: actuarial elements (the severity X, the frequence N, the clauses, the retrocession), financial elements (the financial advantage when claims are paid long after the premium instalment, the remuneration of the shareholders at the cost of capital, the use of a cash flow model), economic elements (inflation, superinflation) and commercial elements (brokerage, administrative expenses).
The Panjer's algorithm is a powerful tool we often use (in fact as many times as there are periods between claims payments in our model) in order to find the aggregate situation of the treaty in the future. Obviously this has a computing cost which is really low nowadays.
The notion of cost of capital has been used in order to provide a fair price for the shareholders.
A lot of parameters are necessary in order to run our model. Note that these parameters would also be necessary within a simplified model. In case some parameters are difficult to estimate, our methodology provides a solution in the sense that it allows easily for sensitivity analyses.
In that respect, it is also worth noting that the sensitivity analyses allow to determine which parameters are sensitive and which are not. For example, we noticed in our numerical application that

- a similar change on rates (interest rates, inflation, ...) has not a big influence.
- the variable first year of premium adjustment seems to be extremely sensitive.

However the former variables are usually studied in detail whereas the latter is classicaly neglected by reinsurers. Our methodology shows which parameters are important.

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