

Economic Value Added Optimization for Insurers Using a Multivariate Student-t Model

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Abstract. In this paper we study the impact of distributions with heavy tails and tail dependence on the underwriting policy of an insurer. The aim is to maximize the value creation by varying the number of contracts an insurer sells, under the constraint that investors provide a given amount of economic capital. To model the underwriting risk, we use a multivariate student-t distribution. We compare the results to those found when using a multivariate normal distribution. We also show that within this model it is possible to write business with a negative margin for particular dependence structures and we consider the impact of a stop loss reinsurance.

Keywords: Required solvency level, Tail value at risk, Risk adjusted capital, Diversification benefit, Economic value added, Stochastic dependence, Multivariate student-t, Tail dependence.

1 Introduction

In this paper we analyze the underwriting policy of an insurance company. An insurer faces many risks. One type of risk is linked to the frequency and severity of the claims. This is called the underwriting risk. Another important risk is the return on the investments. All these risks (underwriting risk, market risk, operational risk,...) have an influence on the value creation. In this paper we focus on the underwriting risk.

Our goal is to determine a portfolio of policies for which the value creation is maximal, given a set of constraints. In order to be sure that the insurer can pay its claims, the regulator will impose him to hold a required capital. This capital cannot be funded with the premium income alone. An important part is provided by the shareholders, who want to be rewarded for their investments. We assume the capital has a cost k .

Creating a positive economic value added (EVA) is not always possible. A lot of factors can influence the EVA. One of them is the distribution of the risks. To model the risks or claims, we will use a multivariate student-t distribution. This allows us to work with fatter tails than those of the normal distribution. The thickness of the tails depends on the number of degrees of freedom which can be chosen as a parameter. Moreover, the multivariate student-t distribution allows to

model tail dependence whereas the tail dependence for the multivariate normal distribution equals zero.

Both the normal and the student-t distributions belong to the class of elliptical distributions. Since Landsman and Valdez (2003) provide us with an analytical form of the allocation formula (proposed by Denault (2001)) for elliptical distributions, these distributions are convenient when solving the maximization problem.

In the applications we show that in some cases the optimal underwriting strategy contains business with a negative margin. Hereby we emphasize that one should always evaluate the global EVA. Decisions based on the EVA of an individual LOB may destroy part of the global EVA of the company.

In sections 2 and 3 we present some well known measures for calculating capital requirements as well as measures for value creation. We also explain how the capital provided by the investors can be allocated to several insurance branches. A short overview of the class of elliptical distributions is given in section 4, where we also present how we model the underwriting risk. In the following section we give some numerical applications. We analyze the effect of stop-loss reinsurance on the value creation in section 6. Section 7 concludes.

2 Risk Measures and Value Creation

We denote the total amount of the claims on an insurance portfolio by a random variable S . The individual risks X_i are modelled by a multivariate random vector \mathbf{X} . We have that $S = X_1 + \dots + X_n$. To pay the claims the insurer receives a total premium income P from its clients. To ensure that the insurance company can pay its claims with a sufficiently high probability, the regulator demands the insurer to hold a certain amount of money, called the required solvency level RSL . The RSL consists of the premium P and a certain amount of capital provided by the shareholders. We call this the risk adjusted capital (RAC). So we have

$$RSL = P + RAC. \quad (1)$$

There are several possibilities to determine the RSL . If we take the RSL equal to the Value at Risk with a given probability q :

$$VaR_q(S) = \inf\{s \in \mathbb{R} | F_S(s) \geq q\}, \quad 0 < q < 1,$$

we have a probability of $1-q$ that $S > VaR_q(S)$. A drawback of the VaR is that it does not take into account the situation

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once $S > VaR_q(S)$. Another popular risk measure is the Tail Value at Risk at level q :

$$TVaR_q(S) = \frac{1}{1-q} \int_q^1 VaR_p(S) dp, \quad 0 < q < 1.$$

Here an average of all the quantiles exceeding the VaR_q is taken, so more information about the upper tail is taken into account. Moreover the $TVaR$ is a subadditive risk measure whereas the VaR is not. This means that for two lines of business the following property holds

$$TVaR_q(S_1 + S_2) \leq TVaR_q(S_1) + TVaR_q(S_2).$$

This property can be motivated, based on the following arguments. In Dhaene (2004a) it is explained why many people want a risk measure to be subadditive. In general a risk measure ρ is called subadditive if for two risks X_1, X_2

$$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2).$$

If we assume that the solvency capital requirement imposed by the regulator is given by ρ , the equation above can be explained as follows: for a risk measure ρ , the capital requirement $\rho(X_1 + X_2)$ for the merged risks is smaller than for the added capitals $\rho(X_1)$ and $\rho(X_2)$. There are some arguments to defend this statement. We therefore first have a look at the shortfall of a risk X defined as

$$\max(0, X - \rho(X)) = (X - \rho(X))_+.$$

If two risks X_1, X_2 are merged, the shortfall decreases because

$$(X_1 + X_2 - \rho(X_1) - \rho(X_2))_+ \leq \sum_{i=1}^2 (X_i - \rho(X_i))_+. \quad (2)$$

This is due to the diversification effect. The loss on one risk can be compensated by a gain on the other. Because of the decrease in shortfall, it is justified to require a less severe capital requirement for the merged risks. From the viewpoint of the regulator the smaller the shortfall or the higher the capital, the better. On the other hand the capital has a cost. We have with probability 1 that

$$(\rho(X_1) + \rho(X_2) - X_1 - X_2)_+ \leq \sum_{i=1}^2 (\rho(X_i) - X_i)_+. \quad (3)$$

The shareholder wants the left hand side of equation (3) as small as possible, which is together with equation (2) a motivation to allow for some subadditivity of ρ . We should of course be careful that ρ is not too subadditive. For more details we refer to Dhaene (2004a).

Henceforth we will use the $TVaR$. Besides the subadditivity property, this risk measure is more specific than the VaR . Two distributions can have the same VaR but a substantially different tail. This can be measured with the $TVaR$. Equation (1) then becomes

$$TVaR_q(S) = P + RAC.$$

As mentioned above, the RAC is the capital provided by the shareholders who want to be rewarded for their investments. We define the economic value added (EVA) as

$$EVA = Margin - kRAC,$$

where the Margin is the loading on top of the pure premium $E[S]$ and k is the cost of capital. The return on the invested capital can be measured by

$$RORAC = \frac{Margin}{RAC} = \frac{EVA}{RAC} + k,$$

where RORAC stands for "return on RAC".

3 Capital Allocation

An insurance company can write several branches S_1, \dots, S_n . The capital provided by the investors is given by

$$RAC = TVaR_q(S) - P,$$

where $S = S_1 + \dots + S_n$. Let P_i denote the premium for line of business i (LOB_i). To distribute the capital over the different lines of business (LOB 's) of the company, we use the following allocation rule (see also Denault (2001) and Kalkbrenner (2005)):

$$RAC_i = E[S_i | S > VaR_q(S)] - P_i, \quad (4)$$

for a each line of business (LOB). This principle satisfies the no undercut axiom (i.e. one of the axioms presented by Denault (2001) to have a *coherent* allocation principle). The total premium is given by the sum

$$P = P_1 + \dots + P_n.$$

It is clear that this allocation principle is additive, i.e.:

$$RAC = RAC_1 + \dots + RAC_n.$$

We define the value creation per LOB as:

$$EVA_i = Margin_i - kRAC_i.$$

We will investigate how many contracts an insurer can sell for the different LOB 's such that the total EVA is maximal for a given RAC . If we want to calculate the expectation in equation (4) we have to know the distribution of S . A first possibility is to assume a normal distribution for S . In order to allow for fatter tails, we will also consider a student-t distribution, which is, just as the normal distribution, a member of the class of elliptical distributions. In the next section we give some notions about this class of distributions. For more details, we refer to Landsman and Valdez (2003).

4 Building the Model

4.1 Elliptical Distributions

Let $\mathbf{X} = (X_1, \dots, X_n)^T$ be a n -dimensional random vector. Let $\boldsymbol{\mu}$ be a column vector and $\boldsymbol{\Sigma}$ a $n \times n$ positive-definite matrix. We say that \mathbf{X} has a multivariate elliptical distribution

$$\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n),$$

if the function g_n satisfies the condition

$$\int_0^\infty x^{\frac{n}{2}-1} g_n(x) dx < \infty. \quad (5)$$

If the density $f_{\mathbf{X}}(\mathbf{x})$ exists, it can be written as

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\Sigma|}} g_n \left[\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right],$$

where c_n is a normalizing constant such that

$$\int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = 1.$$

The function g_n is called the density generator and Σ is called the dispersion matrix.

The following properties will be useful (see Landsman and Valdez (2003)).

Property 4.1. If $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, g_n)$ and A is some $m \times n$ matrix of rank $m \leq n$ and \mathbf{b} is some m -dimensional column vector, then $A\mathbf{X} + \mathbf{b}$ is distributed as

$$A\mathbf{X} + \mathbf{b} \sim E_m(A\boldsymbol{\mu} + \mathbf{b}, A\Sigma A^T, g_m)$$

If we take A equal to $\mathbf{e}^T = (1, \dots, 1)$ we immediately find the following property.

Property 4.2. Suppose $(X_1, \dots, X_n)^T = \mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, g_n)$. If we define the sum $S = X_1 + \dots + X_n$ as $\mathbf{e}^T \mathbf{X}$, where $\mathbf{e} = (1, \dots, 1)^T$ is n -dimensional, then S is also elliptically distributed:

$$S \sim E_1(\mathbf{e}^T \boldsymbol{\mu}, \mathbf{e}^T \Sigma \mathbf{e}, g_1).$$

We now take a look at some specific elliptical distributions.

Normal Distribution

If we choose the density generator $g_n(u)$ equal to e^{-u} , we say that $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, g_n)$ is normally distributed and denote $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \Sigma)$. The density function can then be written as

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\Sigma|}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}.$$

Note that, in the normal case, Σ is equal to $Cov(\mathbf{X})$ (see Landsman and Valdez (2003)).

Student-t Distribution

If the density generator is equal to

$$g_n(u) = \left(1 + \frac{u}{k_p}\right)^{-p},$$

we say that $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, g_n)$ is student-t distributed with parameters $p > n/2$ and k_p . The density function can then be written as

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\Sigma|}} \left(1 + \frac{(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})}{2k_p}\right)^{-p}.$$

Here, $\Sigma = Cov(\mathbf{X})$ does not hold. (In Landsman and Valdez (2003) you find that $\Sigma = -\phi'(0)Cov(\mathbf{X})$, where ϕ is the characteristic generator.)

If we take the parameters $k_p = \frac{m}{2}$ and $p = \frac{n+m}{2}$ (m, n integer), we find the classical student-t distribution with m degrees of freedom. The density function then becomes

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\Sigma|}} \left(1 + \frac{(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})}{m}\right)^{-\frac{m+n}{2}}$$

and the variance-covariance matrix of \mathbf{X} is equal to $Cov(\mathbf{X}) = \frac{m}{m-2}\Sigma$ (see Demarta and McNeil (2005)).

4.2 The Model

The model presented here is a generalization of the model used in Walhin (2005). For multivariate elliptical distributions the following theorem is proven in Landsman and Valdez (2003):

Theorem 4.3.

Let $\mathbf{X} = (S_1, \dots, S_n)^T \sim E_n(\boldsymbol{\mu}, \Sigma, g_n)$ and let $S = S_1 + \dots + S_n$ be $E_1(\mathbf{e}^T \boldsymbol{\mu}, \mathbf{e}^T \Sigma \mathbf{e}, g_1)$ -distributed, such that g_1 satisfies the condition

$$\int_0^\infty g_1(x) dx < \infty.$$

Let $S = S_1 + \dots + S_n$. Then the tail conditional expectation can be written as

$$\mathbb{E}[S_i | S > VaR_q(S)] = \mu_i + \eta_S \sigma_i \sigma_S \rho_{S_i, S},$$

with

$$\eta_S = \frac{\frac{1}{\sigma_S} \bar{G}(\frac{1}{2} z_{S,q}^2)}{\bar{F}_Z(z_{S,q})},$$

$$z_{S,q} = \frac{s_q - \mu_S}{\sigma_S},$$

$$s_q = VaR_q(S),$$

$$G(x) = c_1 \int_0^x g_1(u) du,$$

$$\bar{G}(x) = G(\infty) - G(x),$$

$$F_Z(z) = c_1 \int_{-\infty}^z g_1(\frac{1}{2} x^2) dx,$$

$$\bar{F}_Z(z) = 1 - F_Z(z).$$

If we substitute this result into (4) we get a method to divide the capital over the different lines of business:

$$\begin{aligned} RAC_i &= \mathbb{E}[S_i | S > VaR_q(S)] - P_i \\ &= \mu_{S_i} + \eta_S \sigma_{S_i} \sigma_S \rho_{S_i, S} - P_i \\ &= \eta_S \sigma_{S_i} \sigma_S \rho_{S_i, S} - Margin_i. \end{aligned} \quad (6)$$

Assume that an insurance company sells policies concerning three kinds of risks:

$$\begin{cases} n_1 & \text{risks of type 1: } X_1, \dots, X_{n_1} \\ n_2 & \text{risks of type 2: } X_{n_1+1}, \dots, X_{n_1+n_2} \\ n_3 & \text{risks of type 3: } X_{n_1+n_2+1}, \dots, X_{n_1+n_2+n_3}. \end{cases} \quad (7)$$

Suppose the $n_1 + n_2 + n_3$ risks are distributed with the following mean-, dispersion- and correlation parameters:

$$\left\{ \begin{array}{ll} \mu_{X_j} = \mu_1, & j = 1, \dots, n_1 \\ \mu_{X_j} = \mu_2, & j = n_1 + 1, \dots, n_1 + n_2 \\ \mu_{X_j} = \mu_3, & j = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3 \\ \sigma_{X_j} = \sigma_1, & j = 1, \dots, n_1 \\ \sigma_{X_j} = \sigma_2, & j = n_1 + 1, \dots, n_1 + n_2 \\ \sigma_{X_j} = \sigma_3, & j = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3 \\ \rho_{X_i, X_j} = \rho_{11}, & i = 1, \dots, n_1 \\ & j = i + 1, \dots, n_1 \\ \rho_{X_i, X_j} = \rho_{22}, & i = n_1 + 1, \dots, n_1 + n_2 \\ & j = i + 1, \dots, n_1 + n_2 \\ \rho_{X_i, X_j} = \rho_{33}, & i = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3 \\ & j = i + 1, \dots, n_1 + n_2 + n_3 \\ \rho_{X_i, X_j} = \rho_{12}, & i = 1, \dots, n_1 \\ & j = n_1 + 1, \dots, n_1 + n_2 \\ \rho_{X_i, X_j} = \rho_{13}, & i = 1, \dots, n_1 \\ & j = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3 \\ \rho_{X_i, X_j} = \rho_{23}, & i = n_1 + 1, \dots, n_1 + n_2 \\ & j = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3 \end{array} \right. \quad (8)$$

We define the dispersion matrix Σ as the $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ matrix with elements

$$\begin{aligned} \Sigma(j, j) &= \sigma_{X_j}^2, \quad 1 \leq j \leq n_1 + n_2 + n_3, \\ \Sigma(i, j) &= \rho_{X_i, X_j} \sigma_{X_i} \sigma_{X_j}, \quad 1 \leq i \neq j \leq n_1 + n_2 + n_3. \end{aligned}$$

If we take for example $n_1 = n_2 = n_3 = 2$, Σ equals

$$\begin{pmatrix} \sigma_1^2 & \rho_{11}\sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{11}\sigma_1^2 & \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{22}\sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{22}\sigma_2^2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 & \rho_{33}\sigma_3^2 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{33}\sigma_3^2 & \sigma_3^2 \end{pmatrix} \quad (9)$$

The matrix Σ has to be positive semi-definite. It is shown in Walhin (2005) that the following conditions are sufficient for positive semi-definiteness:

$$\begin{aligned} \rho_{11}\rho_{22}\rho_{33} + 2\rho_{12}\rho_{13}\rho_{23} &\geq \rho_{11}\rho_{23}^2 + \rho_{22}\rho_{13}^2 + \rho_{33}\rho_{12}^2 \\ \rho_{11}\rho_{22} &\geq \rho_{12}^2 \\ \rho_{11}\rho_{33} &\geq \rho_{13}^2 \\ \rho_{22}\rho_{33} &\geq \rho_{23}^2 \\ \rho_{11} &\geq 0 \\ \rho_{22} &\geq 0 \\ \rho_{33} &\geq 0. \end{aligned} \quad (10)$$

Let

$$\mathbf{X} = (X_1, \dots, X_{n_1}, X_{n_1+1}, \dots, X_{n_1+n_2}, X_{n_1+n_2+1}, \dots, X_{n_1+n_2+n_3})^T$$

and

$$S = \underbrace{X_1 + \dots + X_{n_1}}_{S_1} + \underbrace{X_{n_1+1} + \dots + X_{n_1+n_2}}_{S_2} + \underbrace{X_{n_1+n_2+1} + \dots + X_{n_1+n_2+n_3}}_{S_3}$$

with the X_i 's distributed as in (7) and (8). Using Theorem 4.2 we have that $S_k \sim E_1(\mu_{S_k}, \sigma_{S_k}^2, g_1)$ for all $k \in \{1, 2, 3\}$ where

$$\begin{aligned} \mu_{S_k} &= n_k \mu_k \\ \sigma_{S_k}^2 &= n_k \sigma_k^2 + n_k(n_k - 1) \rho_{kk} \sigma_k^2 \end{aligned}$$

and also that $S \sim E_1(\mu_S, \sigma_S^2, g_1)$ where

$$\begin{aligned} \mu_S &= \sum_{k=1}^3 n_k \mu_k \\ \sigma_S^2 &= \sum_{k=1}^3 (n_k \sigma_k^2 + n_k(n_k - 1) \rho_{kk} \sigma_k^2) + 2n_1 n_2 \rho_{1,2} \sigma_1 \sigma_2 \\ &\quad + 2n_1 n_3 \rho_{1,3} \sigma_1 \sigma_3 + 2n_2 n_3 \rho_{2,3} \sigma_2 \sigma_3. \end{aligned}$$

Note that σ_S^2 is not necessarily equal to the variance of S . Finally, we define the premium as

$$P = \sum_{k=1}^3 (1 + \alpha_k) n_k \mu_k,$$

so that

$$Margin = \sum_{k=1}^3 \alpha_k n_k \mu_k.$$

We have now all the information to calculate the EVA.

4.3 Tail Dependence

It is also useful to know how strong the dependence between the tails of the marginal distributions is. Consider two risks X_i, X_j . If X_i is large and the tail dependence between the marginal distributions of X_i and X_j is high, then the probability that X_j is also large, is higher than when no tail dependence is present. We can measure this probability with tail dependence coefficients (see Embrechts et al. (2002)). The upper and lower tail dependence coefficient are given by

$$\begin{cases} \lambda_u = \lim_{q \rightarrow 1} P(X_2 > F_2^{-1}(q) | X_1 > F_1^{-1}(q)) \\ \lambda_l = \lim_{q \rightarrow 0} P(X_2 \leq F_2^{-1}(q) | X_1 \leq F_1^{-1}(q)), \end{cases}$$

if these limits $\lambda_u, \lambda_l \in [0, 1]$ exist and where F_i is the marginal distribution function of X_i . When both coefficients are equal, we use one notation λ . For a bivariate student-t distributed random variable with m degrees of freedom it is proven in Embrechts et al. (2002) that the tail dependence coefficient λ can be written as

$$\lambda = 2F_{t, m+1} \left(-\frac{\sqrt{m+1}\sqrt{1-\rho}}{\sqrt{1+\rho}} \right), \quad (11)$$

where ρ is the correlation between the two components of the bivariate student-t and $F_{t,m+1}$ the standard student-t distribution function.

In table 1 the tail dependence coefficients for different values of m and ρ are given. Note that for the normal distribution $\lambda = 0$. There is no tail dependence in that case. We see that λ increases for an increasing ρ and decreasing m .

m	ρ				
	-0.02	-0.01	0	0.01	0.1
3	0.111	0.113	0.116	0.119	0.145
4	0.071	0.073	0.076	0.078	0.099
5	0.047	0.048	0.050	0.052	0.069
6	0.031	0.032	0.033	0.034	0.048
7	0.020	0.021	0.022	0.023	0.034
8	0.014	0.014	0.015	0.016	0.024
9	0.009	0.010	0.010	0.011	0.017
10	0.006	0.006	0.007	0.007	0.012
15	0.001	0.001	0.001	0.001	0.002

Table 1. Tail dependence coefficients.

5 Numerical Applications

Our goal is to maximize the EVA by changing the number of contracts n_1, n_2, n_3 with a capital constraint. One can easily extend the calculations to a higher number of LOB's. Unless explicitly mentioned otherwise, we maximize the EVA with n_1, n_2, n_3 being non-negative integers under the constraint that the RAC is smaller than or equal to 100. We will investigate several cases with different parameter sets and distributions. We start from the following basic parameters which are also used in Walhin (2005):

$$\begin{aligned}
\mu_1 = \mu_2 = \mu_3 &= 1 & (12) \\
\sigma_1 = \sigma_2 = \sigma_3 &= 1 \\
\rho_{1,1} = \rho_{2,2} = \rho_{3,3} &= 0.1 \\
\rho_{1,2} &= -0.01 \\
\rho_{1,3} &= -0.01 \\
\rho_{2,3} &= 0.01 \\
\alpha_1 = \alpha_2 = \alpha_3 &= 0.1 \\
q &= 0.99 \\
\text{cost of capital} &= 0.15 \\
RAC &= 100
\end{aligned}$$

5.1 Normal Distribution

LOB	n	RAC	Margin	EVA	RORAC
1	94	37.09	9.4	3.84	25.34%
2	79	30.95	7.9	3.26	25.53%
3	80	31.75	8	3.24	25.20%
Total	253	99.79	25.3	10.33	25.35%

Table 2. Optimal portfolio for basic parameters (12).

Note that these results are already obtained in Walhin (2005). We copy the tables to facilitate comparisons with the student-t distribution. For the same reason we always follow the same structure. Here we assume that $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. For the basic

parameters (12) we show them in table 2.

We now create a portfolio where the margin of LOB_1 is negative and for which the risks in LOB_1 are negatively correlated with the risks in LOB_2 and LOB_3 by changing α_1 to -0.01 and $\rho_{1,2}$ and $\rho_{1,3}$ to -0.02 in (12). The results are given in table 3.

LOB	n	RAC	Margin	EVA	RORAC
1	24	-0.37	-0.24	-0.19	66.72%
2	93	50.16	9.3	1.78	18.54%
3	93	50.16	9.3	1.78	18.54%
Total	210	99.96	18.36	3.37	18.37%

Table 3. Change in (12) $\alpha_1 = -0.01$ and $\rho_{1,2} = \rho_{1,3} = -0.02$.

We see that the EVA has decreased. For LOB_1 the EVA is even negative, so we could think about dropping this branch. We then find:

LOB	n	RAC	Margin	EVA	RORAC
1	0	0	0	0	.
2	90	49.76	9	1.54	18.09%
3	90	49.76	9	1.54	18.09%
Total	180	99.51	18	3.07	18.09%

Table 4. Abandoning LOB_1 .

This result tells us that it is better to keep LOB_1 even when the business itself creates a negative EVA_1 due to the negative margin. Because of the negative correlation with LOB_2 and LOB_3 the total EVA is higher than without LOB_1 . This is an excellent example of diversification of the portfolio. Because RAC_1 is negative, the other branches can consume more capital than when LOB_1 is dropped.

If we start again from the basic parameters given in (12) but double the RAC to 200 we find an optimal portfolio as given in table 5.

LOB	n	RAC	Margin	EVA	RORAC
1	194	73.54	19.4	8.37	26.38%
2	165	63.06	16.5	7.04	26.17%
3	165	63.06	16.5	7.04	26.17%
Total	524	199.66	52.4	22.45	26.24%

Table 5. Doubled initial capital.

The EVA has more than doubled. So due to the diversification, there is a relative gain. The more risks you write, the less capital they individually consume.

Changing the parameters of the first n_1 risks (starting from (12)) into $\mu_1 = 2$ and $\sigma_1 = 2$ leads to the results shown in table 6. We find a decrease in the EVA because the risks of type 1 are larger than before. Those risks consume more capital and there is less diversification.

5.2 Student-t Distribution

As already mentioned above, the normal distribution does not have large tails. It is therefore interesting to use the student-t distribution with m degrees of freedom to model S . The smaller m , the larger the tails. For m going to infinity we find the normal distribution. In section 4.1 we saw that $Cov(\mathbf{X}) =$

LOB	n	RAC	Margin	EVA	RORAC
1	45	38.35	9	3.25	23.47%
2	78	30.39	7.8	3.24	25.67%
3	79	31.18	7.9	3.22	25.33%
Total	202	99.92	24.7	9.71	24.72%

Table 6. Enlarging the first type of risks.

$\frac{m}{m-2}\Sigma$. We will therefore consider only student-t distributions where $m \geq 3$. Note that a student-t distribution for which the number of degrees of freedom equals 2, leads to an infinite variance. To make the calculations comparable to those of section 5.1, we want the variance-covariance matrix for our student-t distributions to be equal to the one of the normal distribution. Therefore, if we use a student-t distribution with m degrees of freedom, we will multiply the dispersion matrix we used for the normal distribution with a factor $\frac{m-2}{m}$. We denote the dispersion matrix obtained as such with Σ^m . We then find the new basic parameter set as given by (13).

$$\begin{aligned}
\mu_1 = \mu_2 = \mu_3 &= 1 & (13) \\
\sigma'_1 = \sigma'_2 = \sigma'_3 &= \sqrt{\frac{m-2}{m}} \\
\rho_{1,1} = \rho_{2,2} = \rho_{3,3} &= 0.1 \\
\rho_{1,2} &= -0.01 \\
\rho_{1,3} &= -0.01 \\
\rho_{2,3} &= 0.01 \\
\alpha_1 = \alpha_2 = \alpha_3 &= 0.1 \\
q &= 0.99 \\
\text{cost of capital} &= 0.15 \\
RAC &= 100
\end{aligned}$$

From (9) it is now clear that

$$Cov(\mathbf{X}) = \Sigma$$

because $Cov(\mathbf{X}) = \frac{m}{m-2}\Sigma'$ and $\Sigma' = \frac{m-2}{m}\Sigma$.

In the next sections we will calculate the optimal number of contracts to maximize the EVA for different values of m . For the tail dependence coefficients we refer to table 1.

5.2.1 $m=3$

LOB	n	RAC	Margin	EVA	RORAC
1	115	73.47	11.5	0.48	15.65%
2	98	63.23	9.8	0.32	15.50%
3	98	63.23	9.8	0.32	15.50%
Total	311	199.92	31.1	1.11	15.56%

Table 7. Student-t ($m=3$) with double capital.

With the adapted basic parameters (13) it is better to sell no contracts at all. If policies are underwritten anyway, the EVA will be negative.

Changing the parameters α_1 to -0.01 and $\rho_{1,2}$ and $\rho_{1,3}$ to -0.02 also leads to an economic value added of 0. Since n_1 is then zero, LOB_1 is dropped. Of course, enlarging the first n_1 risks by putting $\mu_1 = 2$ and $\sigma'_1 = 2\sqrt{\frac{m-2}{m}}$ makes it only

worse. The EVA remains zero.

Multiplying the RAC with 2 and hence creating more diversification opportunity, we find an optimal strategy in which value can be created by taking the setup given in table 7.

5.2.2 $m=4$

LOB	n	RAC	Margin	EVA	RORAC
1	61	36.59	6.1	0.61	16.67%
2	52	31.49	5.2	0.48	16.51%
3	52	31.49	5.2	0.48	16.51%
Total	165	99.57	16.5	1.56	16.57%

Table 8. Student-t ($m=4$) for the basic parameters (13).

LOB	n	RAC	Margin	EVA	RORAC
1	129	74.29	12.9	1.76	17.36%
2	109	62.65	10.9	1.50	17.40%
3	109	62.65	10.9	1.50	17.40%
Total	347	199.58	34.7	4.76	17.39%

Table 9. Student-t ($m=4$) with double capital.

In table 8 the results are given for the basic parameter set (13). When we change the parameter α_1 to -0.01 and $\rho_{1,2}$ and $\rho_{1,3}$ to -0.02 , it is not possible to create a positive EVA. Dropping LOB_1 doesn't help.

If we double the RAC , we again obtain more than twice the EVA (see table 9).

Starting from (13) and enlarging the risks of type 1 leads to the results as given in table 10.

LOB	n	RAC	Margin	EVA	RORAC
1	28	36.26	5.6	0.16	15.44%
2	52	31.84	5.2	0.42	16.33%
3	52	31.84	5.2	0.42	16.33%
Total	132	99.94	16	1.01	16.01%

Table 10. Student-t ($m=4$) with larger risks in LOB_1 .

5.2.3 Higher Degrees of Freedom

As we have seen, changing α_1 into -0.01 and $\rho_{1,2}$ and $\rho_{1,3}$ into -0.02 in (13) results in an EVA of zero for $m \in \{3, 4\}$. Only from $m = 8$ on, the EVA becomes positive for this parameter set, we then have the portfolio shown in table 11.

LOB	n	RAC	Margin	EVA	RORAC
1	17	-0.49	-0.17	-0.10	34.83%
2	76	49.52	7.6	0.17	15.35%
3	77	50.75	7.7	0.09	15.17%
Total	170	99.78	15.13	0.16	15.16%

Table 11. Student-t ($m=8$) with $\alpha_1 = -0.01$ and $\rho_{1,2} = \rho_{1,3} = -0.02$.

Let us now compare this result with table 12, where we assume the basic parameters (13) and set m equal to 8. We conclude that the EVA has decreased. This was expected because the margin of LOB_1 was made negative. In spite of this negative margin, it remains possible to underwrite these risks and to create a positive EVA due to the negative correlation with the other branches. Moreover, if we drop LOB_1 the EVA becomes zero again.

LOB	n	RAC	Margin	EVA	RORAC
1	76	36.41	7.6	2.14	20.87%
2	65	31.61	6.5	1.76	20.56%
3	65	31.61	6.5	1.76	20.56%
Total	206	99.64	20.6	5.65	20.67%

Table 12. Student-t ($m=8$) with basic parameters (13).

Setting m equal to 7, α_1 to -0.01 and $\rho_{1,2}$ and $\rho_{1,3}$ to -0.02 results in an EVA of zero. As before, omitting LOB_1 is not an interesting option. But if we raise the capital to 500 we can get a positive economic value added (see table 13).

We can conclude that it is possible to underwrite risks with a negative margin due to the negative dependence with the other risks.

LOB	n	RAC	Margin	EVA	RORAC
1	110	-6.44	-1.1	-0.13	17.08%
2	395	253.10	39.5	1.54	15.61%
3	395	253.10	39.5	1.54	15.61%
Total	900	499.75	77.9	2.94	15.59%

Table 13. Student-t ($m=7$) with $RAC = 500$.

5.2.4 Summary

In table 14 we present the values for the EVA for the different cases we discussed:

- case 1: basic parameters (12) or (13).
- case 2: let $\alpha_1 = -0.01$ and $\rho_{1,2} = \rho_{1,3} = -0.02$ in case 1.
- case 3: omit LOB_1 in case 2.
- case 4: let $RAC = 500$ in case 2.
- case 5: let $RAC = 200$ in case 1.
- case 6: double μ_1 and σ_1 in case 1.

We have put the EVA in bold and the optimal numbers of contracts (n_1 n_2 n_3) between brackets.

The lower m , the fatter the tails. It is therefore logical that the EVA decreases as m decreases. Compared to the normal distribution, the student-t distribution results in a lower value creation. The conclusions for the different cases are the same for both distributions. Furthermore we remark that the larger the tails are, the stronger the relative effect of changing a parameter is. From case 1 to case 2 for example, the EVA decreases to 30% of the initial value for the normal distribution. Considering the same change but for the student-t with $m = 9$, results in a decrease to 8%. The same reasoning can be made for other changes. We conclude that the distribution of the underwriting risk plays an important role.

If we have a look at table 1, we see that λ depends on m and ρ . The EVA also changes for different values of m and ρ . However, the effect of the tail dependence on the EVA is difficult to analyze. In general an increasing λ leads to a decrease of the EVA.

In table 15 we calculate the optimal portfolio with the following non-symmetric parameters: $\mu_1 = 2$, $\mu_2 = 3$, $\mu_3 = 4$;

$$\sigma'_1 = \sqrt{\frac{m-2}{m}}, \sigma'_2 = 2\sqrt{\frac{m-2}{m}}, \sigma'_3 = 3\sqrt{\frac{m-2}{m}}; \rho_{1,1} = 0.05, \rho_{2,2} = 0.1, \rho_{3,3} = 0.15, \rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.01; \alpha_1 = \alpha_2 = \alpha_3 = 0.1; q = 0.99, k = 0.15 \text{ and } RAC = 100.$$

LOB	n	RAC	Margin	EVA	RORAC
1	207	75.00	41.4	30.15	0.55%
2	28	16.88	8.4	5.87	0.50%
3	9	7.95	3.6	2.41	0.45%
Total	244	99.83	53.4	38.42	0.53%

Table 15. Optimal portfolio with non symmetric parameters.

The risks of LOB_3 are larger than those of the other LOB 's. However, it remains interesting to underwrite risks of LOB_3 . Abandoning LOB_3 results in a lower EVA. We conclude that local decisions are not appropriate.

To end this section, we discuss equation (6)

$$RAC_i = \eta_S \sigma_i \rho_S \rho_{S_i,S} - Margin_i.$$

In table 11 we see that $RAC_1 < 0$ because ρ_{12} and ρ_{13} were set to -0.02 . Having $\rho_{S_i,S} < 0$ is not necessary to make RAC_i negative. We can also take a high margin. An example is shown in table 16, where all ρ_{ij} equal zero. For the other parameters we set μ_i and σ_i equal to 1, α_i equal to 0.1 and m equal to 4. The results are given for a portfolio with 500 contracts for each LOB . An optimal portfolio does not exist in this case: the more contracts you underwrite, the more value you create. All the capital is provided by the margin.

LOB	n	RAC	Margin	EVA	RORAC
1	500	-2.34	50	50.35	-21.35%
2	500	-2.34	50	50.35	-21.35%
3	500	-2.34	50	50.35	-21.35%
Total	1500	-7.03	150	151.05	-21.35%

Table 16. Portfolio with negative RAC_i .

6 Reinsurance

In this section we analyze the effect of an unlimited stop-loss reinsurance with a threshold d . The retention for the cedent is then $\min(S, d)$, so less capital is required. Taking the same RAC allows for more risks to be underwritten. On the other hand a reinsurance premium has to be paid. We investigate the effects on the economic value added. We assume that there is no credit risk, i.e. that the reinsurer never defaults.

The pure reinsurance premium is defined as

$$\begin{aligned} PP^{Re} &= E[(S - d)_+] \\ &= \int_d^\infty (1 - F_S(s)) ds \\ &= \sigma_S \int_{\frac{d-\mu_S}{\sigma_S}}^\infty (1 - F_Y(y)) dy \end{aligned} \quad (14)$$

where we set $\frac{S-\mu_S}{\sigma_S}$ equal to Y . The commercial premium P^{Re} contains a loading β , such that $P^{Re} = (1 + \beta)PP^{Re}$. The retention is denoted as S^{Ret} . In the future we will use the

Distribution	Cases					
	1	2	3	4	5	6
STUD-T						
$m = 3$	0 (0 0 0)	0 (0 0 0)	0 (0 0 0)	0 (0 98 98)	1.11 (115 98 98)	0 (0 0 0)
$m = 4$	1.56 (61 52 52)	0 (0 0 0)	0 (0 0 0)	0 (0 0 0)	4.76 (129 109 109)	1.01 (28 52 52)
$m = 5$	3.07 (67 56 57)	0 (0 0 0)	0 (0 0 0)	0 (0 0 0)	7.82 (140 119 119)	2.51 (31 56 57)
$m = 6$	4.20 (71 60 61)	0 (0 0 0)	0 (0 0 0)	0.007 (107 308 308)	10.05 (148 126 126)	3.61 (33 60 60)
$m = 7$	5.03 (74 63 63)	0 (0 0 0)	0 (0 0 0)	2.94 (111 395 395)	11.74 (155 131 131)	4.43 (35 62 62)
$m = 8$	5.65 (76 65 65)	0.16 (17 76 77)	0 (0 0 0)	5.19 (115 406 407)	13.04 (160 135 135)	5.05 (36 64 64)
$m = 9$	6.16 (78 66 67)	0.52 (19 78 79)	0.31 (0 76 77)	6.96 (117 415 416)	14.06 (164 138 138)	5.56 (37 65 66)
$m = 10$	6.56 (80 67 68)	0.80 (21 80 80)	0.58 (0 77 78)	8.39 (120 422 423)	14.91 (167 141 141)	5.96 (38 66 67)
$m = 15$	7.83 (85 71 72)	1.65 (18 84 84)	1.42 (0 82 82)	12.71 (126 444 445)	17.42 (176 149 149)	7.22 (40 71 71)
NORMAL	10.33 (94 79 80)	3.37 (24 93 93)	3.07 (0 90 90)	21.37 (143 489 489)	22.45 (194 165 165)	9.71 (45 78 79)

Table 14. Optimal portfolio and maximal EVA for different distributions and cases.

threshold $d = VaR_q(S) = VaR_{99\%}(S)$. The formulas for S^{Ret} are a little different from those for S . We define them below

$$\begin{aligned}
VaR_{99\%}(S^{Ret}) &= d \\
TVaR_{99\%}(S^{Ret}) &= d \\
RAC(S^{Ret}) &= d - (P - P^{Re}) \\
Margin(S^{Ret}) &= Margin(S) - \beta PP^{Re} \\
EVA(S^{Ret}) &= Margin(S^{Ret}) - kRAC(S^{Ret}) \\
RORAC(S^{Ret}) &= \frac{Margin(S^{Ret})}{RAC(S^{Ret})}.
\end{aligned}$$

We recalculate the optimal portfolio (n_1, n_2, n_3) , but maximize the EVA for S^{Ret} instead of the EVA for S . We will not analyze the value creation and the RORAC for each line of business because we consider a stop-loss reinsurance on the whole portfolio. As before we give the results for the different distributions.

6.1 Normal Distribution

The pure reinsurance premium PP^{Re} for this distribution can be written as

$$PP^{Re} = \sigma_S \phi\left(\frac{d - \mu_S}{\sigma_S}\right) - (d - \mu_S) \left[1 - \Phi\left(\frac{d - \mu_S}{\sigma_S}\right)\right].$$

In table 17 (see Walhin (2005)) we find the optimal underwriting policy for the basic parameters (12) and different values of the loading β . We see that buying reinsurance allows to increase the optimal number of contracts for all lines of business, even when the commercial premium is twenty times higher than the pure premium. In the next section we consider a student-t based model.

β	n_1	n_2	n_3	d	RAC	EVA	RORAC
no	94	79	80	362	99.79	10.33	25.35%
1	113	95	95	433	99.72	15.15	30.20%
4	112	95	95	431	99.98	14.45	29.45%
9	111	94	94	427	99.97	13.23	28.23%
14	110	93	93	423	99.95	12.02	27.03%
19	109	92	92	419	99.91	10.84	25.85%

Table 17. Optimal portfolio with reinsurance for normal distribution.

6.2 Student-t Distribution

6.2.1 $m=3$

The lower m , the larger the tails. The table with the results for reinsurance is given below for $m = 3$. We have taken the basic parameter set (13).

β	n_1	n_2	n_3	d	RAC	EVA	RORAC
no	0	0	0	0	0	0	0
1	95	80	80	379	99.81	9.86	24.88%
4	93	78	79	372	99.96	7.37	22.37%
9	89	76	76	359	99.78	3.40	18.41%
14	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0

Table 18. Optimal portfolio with reinsurance for student-t distribution ($m=3$).

This is an interesting result. We see that without buying a reinsurance cover, no value can be created so it is better not to sell any contracts. But now, reinsurance allows to underwrite risks and to create value. However the stop-loss premium cannot be too high, otherwise the EVA again becomes zero.

6.2.2 $m=4$

A positive EVA can be created without buying reinsurance, but the EVA with reinsurance is much higher as long as the reinsurance premium is not unreasonably high.

β	n_1	n_2	n_3	d	RAC	EVA	RORAC
no	61	52	52	248	99.57	1.56	16.57%
1	94	79	79	376	99.66	9.76	24.80%
4	93	78	78	371	100	7.97	22.97%
9	90	76	76	361	99.71	5.02	20.04%
14	88	74	74	352	99.70	2.23	17.24%
19	0	0	0	0	0	0	0

Table 19. Optimal portfolio with reinsurance for student-t distribution ($m=4$).

The same conclusions can be made for higher degrees of freedom. The EVA will become higher and higher because the tails shrink as m increases.

7 Conclusion

In this paper we made a lot of EVA-calculations for different parameter sets within a multivariate student-t framework. Similar calculations can be made for other elliptical distributions like the logistic and exponential power distribution. Our goal was to maximize the EVA under a capital constraint by varying the number of contracts underwritten. To model the total claim amount, we used a multivariate normal or a multivariate student-t distribution. When the distribution for the claims has large tails, it may be more appropriate to choose a student-t distribution. The thickness of the tails increases with a decreasing number of degrees of freedom. Due to their symmetry, elliptical distributions may not always be the most adequate to model insurance risk. Logelliptical distributions may be an interesting generalization in this context. See e.g. Vanduffel and Dhaene (2006) and Valdez and Dhaene (2005).

We saw that under a model with the same variance-covariance matrix, the EVA is lower for a multivariate student-t distribution than for a multivariate normal distribution. Within the class of student-t distributions, the EVA decreases with a decreasing number of degrees of freedom. These effects are due both to the size of the tails and the tail dependence. For multivariate student-t distributions, the tail dependence is larger for a lower number of degrees of freedom. For multivariate normal distributions, the tail dependence is zero.

The following conclusions which were made by Walhin (2005) for the normal case also hold for the student-t distribution. Firstly, when the capital is doubled, the number of contracts underwritten and the EVA are more than doubled. Secondly, when the risks are enlarged, the EVA decreases. Finally, risks with negative correlation have a positive impact on the EVA, even when the margin is negative. This shows it is important to evaluate the portfolio as a whole. In particular,

making decisions on the basis of local EVA's may lead to situations where value is destroyed. Therefore only the global EVA makes sense.

In the last section, we have shown that stop-loss reinsurance has a positive effect on the EVA as long as the reinsurance premium is not unreasonably high.

Finally, we note that our model only takes into account the underwriting risk and the cost of the capital. Other risks and expenses may have an impact on the EVA.

References

- [1] S. Demarta and A.J. McNeil. (2005) The t copula and related copulas. *International Statistical Review*, 1. 111-129.
- [2] M. Denault. (2001) Coherent Allocation of Risk Capital. *Journal of Risk*, 4. 1-33.
- [3] J. Dhaene, R.J.A. Laeven, S. Vanduffel, G. Darkiewicz, and M.J. Goovaerts (2004a), Can a coherent risk measure be too subadditive? research report OR 0431.
- [4] J. Dhaene, S. Vanduffel, Q. Tang, M.J. Goovaerts, R. Kaas and D. Vyncke. (2004b) Capital Requirements, risk measures and comonotonicity. *Belgian Actuarial Bulletin*, 4. 53-61.
- [5] P. Embrechts, et al. (2002) Correlation and dependency in risk management: Properties and pitfalls. *Risk Management: Value at Risk and Beyond*, 176-223. Cambridge University Press, Cambridge.
- [6] M. Kalkbrenner. (2005) An axiomatic approach to capital allocation. *Mathematical Finance*, 15(3). 425-437.
- [7] Z. Landsman and E.A. Valdez. (2003) Tail conditional expectations for elliptical distributions. *North American Actuarial Journal*, 7. 55-71.
- [8] Z. Landsman and E.A. Valdez. (2005) Tail conditional variance for elliptically contoured distributions. *Belgian Actuarial Bulletin*, To appear.
- [9] H. Panjer. (2001) Measurement of risk, solvency requirements and allocation of capital within financial conglomerates. Research Report 01-15, Institute of Insurance and Pension Research, University of Waterloo.
- [10] E. Valdez. and J. Dhaene. (2005) Convex order bounds for sums of dependent log-elliptical random variables. Working paper.
- [11] S. Vanduffel and J. Dhaene (2006). Some results on Denault's capital allocation rule. KUleuven FETEW research report AFI0601.
- [12] J.F. Walhin. (2006) Value creation for insurers. *German Actuarial Bulletin*, 27(4). 681-693.