de Finetti's Retention Problem for Proportional Reinsurance Revisited

F. Glineur † and J.-F. Walhin*•

† *Centre for Operations Research and Econometrics, Université Catholique de Louvain*

∗ *Institut des Sciences Actuarielles, Université Catholique de Louvain*

• *Secura*

Classification code : C61, IM52

Abstract

We use convex optimization to provide a rigorous proof of de Finetti's retention result for proportional reinsurance. We then extend this result to variable quota share reinsurance and surplus reinsurance with table of lines. We observe on a numerical example that neither variable quota share reinsurance nor surplus reinsurance with table of lines may be considered as optimal reinsurance structures.

Corresponding author :

Jean-François Walhin
Secura
Avenue des Nerviens, 9-31
B-1040 Bruxelles
Belgique
phone : + 32 2 504 82 22
fax : + 32 2 504 82 00
e-mail : jfw@secura-re.com

de Finetti's Retention Problem for Proportional Reinsurance Revisited

Abstract

We use convex optimization to provide a rigorous proof of de Finetti's retention result for proportional reinsurance. We then extend this result to variable quota share reinsurance and surplus reinsurance with table of lines. We observe on a numerical example that neither variable quota share reinsurance nor surplus reinsurance with table of lines may be considered as optimal reinsurance structures.

Keywords

Proportional Reinsurance, Quota Share Reinsurance, Surplus Reinsurance, Variable Quota Share Reinsurance, Table of Lines, Convex Optimization.

1 Introduction

Consider a portfolio consisting of n independent risks being subject to possible reinsurance. It is a natural question to look for an optimal reinsurance structure. de Finetti (1940) analyzed the optimality properties of two types of reinsurance : proportional reinsurance and excess of loss reinsurance. For the latter, de Finetti (1940) introduced parametric assumptions about the distributions of the risks whereas his model did not require parametric assumptions in order to analyze the optimality properties of proportional reinsurance.

In the present paper, we will only discuss proportional reinsurance without making any parametric assumption. We will extend the results of de Finetti to other types of proportional reinsurance, namely variable quota share reinsurance and surplus reinsurance with table of lines. Furthermore, as noticed by Bühlmann (1970), the proof of de Finetti's result for quota share reinsurance is complicated. We will introduce adequate tools from convex optimization in order to provide an easy proof for both the original result of de Finetti as well as the proposed extensions. The rest of the paper is organized as follows. Section 2 describes proportional reinsurance. Section 3 describes de Finetti's model and the optimality results for quota share reinsurance. Section 4 introduces the results from convex optimization we will use. In section 5 we prove the optimality result for proportional reinsurance. Sections 6 and 7 are devoted to the optimality results for variable quota share reinsurance and surplus reinsurance with table of lines. Section 8 concludes with a numerical illustration.

2 Proportional Reinsurance

In a situation of proportional reinsurance, the ceding company and the reinsurer agree on a *cession percentage*, say $0 \le a_i \le 1$, for each policy in the portfolio. The premium corresponding to the policy i, say P_i, is shared proportionally between the insurer and the reinsurer.

The reinsurer receives a_iP_i whereas the insurer keeps the premium $(1-a_i)P_i$ (the factor $(1-a_i)$ is called the *retention percentage*). If S_i is a claim hitting policy i, the reinsurer is liable for a_iS_i whereas the insurer retains $(1-a_i)S_i$.

Clearly, the way a proportional reinsurance works is extremely simple. Looking now at the way the cession percentage a_i is chosen, we can distinguish between two types of proportional reinsurance : quota share reinsurance and surplus reinsurance.

2.1 Quota Share Reinsurance

Quota share reinsurance is the easiest way of covering an insurance portfolio. In this case, the cession percentage a_i is the same across the whole insurance portfolio : it will be denoted by a, with $0 \leq a \leq 1$.

2.2 Surplus Reinsurance

In the case of surplus reinsurance, the cession percentage is a function of both the sum insured (SI_i) and a quantity called the line, or retention, chosen by the ceding company.

The line (R) is the maximal amount that the insurer is willing to pay in case of a loss (for each policy in the portfolio). If one wants to make use of proportional reinsurance and ensure that the maximal loss will never exceed the line, one can easily see that the retention percentage must be equal to

$$1-a_i = \min\left(1, \frac{R}{SI_i}\right)$$

which is always between 0 and 1 (since the line is positive). The reason for the min operator is that when the sum insured is smaller then the line, i.e. when $\frac{R}{SI_i}$ is greater than 1, the retention percentage has to be equal to 1.

The cession percentage must be then defined as

$$a_i = 1-\min\left(1, \frac{R}{SI_i}\right) = \max\left(0, 1-\frac{R}{SI_i}\right)$$

with $0 \leq a_i \leq 1$. In the case of a total loss, the retained loss is

$$(1-a_i)SI_i = \min\left(1, \frac{R}{SI_i}\right) \times SI_i = \begin{cases} SI_i \text{ if } SI_i \leq R \\ \frac{R}{SI_i}SI_i = R \text{ if } SI_i \geq R. \end{cases}$$

3 de Finetti's Result for Proportional Reinsurance

Let us consider a portfolio with n independent risks, $S_1, \ldots, S_n$ against premiums P_1, $\ldots$,P_n. Let us assume that each risk may be protected by a proportional reinsurance with cession percentage a_i. Let us also assume that the reinsurer charges a premium according to the *expected value premium principle* with loading ξ_i: the reinsurance premium then writes $(1+\xi_i)a_i\mathbb{E}S_i$.

de Finetti (1940) suggested to choose the cession rates a_i by minimizing the variance of the result of the insurer subject to a given level for the expected result. The result of the insurer is given by

$$Z(\boldsymbol{a}) = \sum_{i=1}^{n} \big(P_i - (1+\xi_i)a_i \mathbb{E}S_i - (1-a_i)S_i\big)$$

where $\boldsymbol{a}$ is the vector of the cession rates : $\boldsymbol{a} = (a_1, \ldots, a_n)$. The cession rates come now from the solution of the following variance minimization programme

$$\min \mathbb{V}ar Z(\boldsymbol{a}) \tag{3.1}$$

subject to

$$\begin{aligned} \mathbb{E}Z(\boldsymbol{a}) &= k && (3.2)\\ a_i &\geq 0 \quad , \quad i = 1, \ldots, n && (3.3)\\ a_i &\leq 1 \quad , \quad i = 1, \ldots, n && (3.4) \end{aligned}$$

where k denotes the expected result chosen by the insurer.

de Finetti (1940) provides the following optimal value for the cession percentages:

$$a_i = \min\left(1, \max\left(0, 1 - \frac{c\xi_i \mathbb{E}S_i}{\mathbb{V}ar S_i}\right)\right) \tag{3.5}$$

where c is a constant that must be computed a posteriori by plugging (3.5) into (3.2). de Finetti's proof is based on results for classical optimization under equality constraints only: he therefore starts by ignoring the inequality constraints (3.3) and (3.4) defining the admissible range for the cession percentages. The solution to the minimization of the variance of $Z(\boldsymbol{a})$ subject to the expected result constraint (3.2) only is then immediately given by

$$a_i = 1 - \frac{c\xi_i \mathbb{E}S_i}{\mathbb{V}ar S_i} \tag{3.6}$$

where c is a Lagrange multiplier introduced for the constraint (3.2). Then, as noticed by Bühlmann (1970), de Finetti (1940) showed, using heavy mathematics, that the value given by (3.5) (which is simply (3.6) adjusted to lie between 0 and 1) is indeed the correct solution to the original minimization programme including the inequality constraints (3.3) and (3.4).

4 Optimality conditions for convex optimization

4.1 Nonlinear optimization

Our simpler proof for de Finetti's result will be based on a basic result from convex optimization which generalizes the notion of Lagrange multipliers to the case of inequality constraints. Consider the following optimization problem with both equality and inequality constraints :

$$\min f(\boldsymbol{x}) \tag{4.1}$$

subject to

$$\begin{aligned} g_j(\boldsymbol{x}) &\leq c_j \quad , \quad j = 1, \ldots, m \\ h_k(\boldsymbol{x}) &= d_k \quad , \quad k = 1, \ldots, p \end{aligned}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$ is a vector of unknowns and functions f, g_j and h_k defined on $\mathbb{R}^n$ define the objective and the constraints. This is a standard formulation of a general nonlinear optimization problem.

Let us introduce the following quantity, called Lagrangian:

$$L(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \sum_{j=1}^{m} \mu_j (g_j(\boldsymbol{x}) - c_j) + \sum_{k=1}^{p} \lambda_k (h_k(\boldsymbol{x}) - d_k)$$

where quantities μ_j ($j = 1, \ldots, m$) and λ_k ($k = 1, \ldots, p$) are Lagrange multipliers for the constraints $g_j(\boldsymbol{x}) \leq c_j$ and $h_k(\boldsymbol{x}) = d_k$ respectively. The following first-order conditions, known as the Karush-Kuhn-Tucker (KKT) conditions, are necessary[1] for optimality of the vector $\boldsymbol{x}$:

$$\begin{aligned} \frac{\partial}{\partial x_i} L(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) &= 0 \quad \text{for } i = 1, \ldots, n \\ \mu_j \geq 0, g_j(\boldsymbol{x}) \leq c_j \text{ and } \mu_j (g_j(\boldsymbol{x}) - c_j) &= 0 \quad \text{for } j = 1, \ldots, m \\ h_k(\boldsymbol{x}) &= d_k \quad \text{for } k = 1, \ldots, p. \end{aligned}$$

(note that when there are no inequality constraints, the second term in the Lagrangian and the second set of constraints both disappear, and we obtain the classical result involving Lagrange multipliers for optimization under equality constraints).

4.2 Convex optimization

An major inconvenient with the above result is that it does not provide a sufficient condition to ensure optimality of a given vector. In order to improve this, we need to introduce the notion of convex function: a function f defined on $\mathbb{R}^n$ is said to be convex if and only if the following inequality holds

$f(\theta\boldsymbol{x} + (1-\theta)\boldsymbol{y}) \leq \theta f(\boldsymbol{x}) + (1-\theta) f(\boldsymbol{y})$ for all θ belonging to the interval $[0\ 1]$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

(the condition means that the graph of the function has to lie below its cords everywhere).

A nonlinear optimization problem of the form (4.1) is said to be convex itself if we can ensure that the objective function f and inequality-defining functions g_j are convex while equality-defining functions h_k are linear. It is known in that case that the KKT optimality conditions are also sufficient, i.e. that any solution[2] to the KKT system is a global minimum

[1] Actually, an additional technical condition known as *constraint qualification* is required for this result to hold, but this will hold trivially for the problems described here, see the reference given below for details.

[2] Note that a given convex problem may admit several optimal solutions, sharing a common objective value and forming a convex set in the feasible domain.

for the minimization problem (4.1). Minimizing the original programme or solving the system of KKT conditions are thus completely equivalent in the case of a convex problem. The reader may for example consult chapter 12 in Nocedal and Wright (1999) for a proof of these results and further details.

5 A Proof Based on Convex optimization for Proportional Reinsurance

In order to prove de Finetti's result about the minimization problem (3.1), we will first show that it is convex, which means that writing and solving its KKT optimality conditions will be enough to find its optimal solutions.

The objective of the minimization programme (3.1) can be written as

$$\mathbb{V}ar Z(\boldsymbol{a}) = \mathbb{V}ar\big[-\sum_{i=1}^{n}(1-a_i)S_i\big] = \sum_{i=1}^{n}(1-a_i)^2\mathbb{V}ar S_i$$

while the expected result can be evaluated as follows

$$\begin{aligned}\mathbb{E}Z(\boldsymbol{a}) &= \sum_{i=1}^{n}\mathbb{E}\big(P_i-(1+\xi_i)a_i\mathbb{E}S_i-(1-a_i)S_i\big)\\ &= \sum_{i=1}^{n}\big(P_i-(1+\xi_i)a_i\mathbb{E}S_i-(1-a_i)\mathbb{E}S_i\big)\\ &= \sum_{i=1}^{n}\big(P_i-\xi_i a_i\mathbb{E}S_i-\mathbb{E}S_i\big).\end{aligned}$$

This leads to the following minimization problem

$$\min\sum_{i=1}^{n}(1-a_i)^2\mathbb{V}ar S_i$$

subject to

$$\begin{aligned}\sum_{i=1}^{n}\xi_i\mathbb{E}S_i a_i &= -k+\sum_{i=1}^{n}P_i-\sum_{i=1}^{n}\mathbb{E}S_i \qquad (5.1)\\ a_i &\leq 1 \quad , \quad i=1,\ldots,n\\ -a_i &\leq 0 \quad , \quad i=1,\ldots,n.\end{aligned}$$

This is a convex optimization programme: indeed, the objective function is convex (as a sum of simple convex quadratic functions), as well as the functions a_i and $-a_i$ that define the inequality constraints (since a linear function is obviously also convex). Moreover, the equality constraint (5.1) are also linear, which implies that the whole problem is convex and therefore that the KKT optimality conditions will be sufficient.

The Lagrangian is

$$L(\boldsymbol{a}, \boldsymbol{y}, \boldsymbol{z}, \lambda) = \sum_{i=1}^{n} \Big[(1-a_i)^2 \mathbb{V}arS_i + y_i(a_i-1) + z_i(-a_i) \Big] + \lambda \Big(\sum_{i=1}^{n} \xi_i \mathbb{E}S_i a_i + k - \sum_{i=1}^{n} P_i + \sum_{i=1}^{n} \mathbb{E}S_i \Big)$$

where we introduced Lagrange multiplier λ for the equality constraints and Lagrange multipliers y_i and z_i for the inequality constraints $a_i \leq 1$ and $-a_i \leq 0$. We can now write down the KKT conditions:

$$\begin{array}{rcll} 2\mathbb{V}arS_i(a_i - 1) + \lambda \xi_i \mathbb{E}S_i + y_i - z_i & = & 0 & i = 1, \dots, n \\ y_i(a_i - 1) & = & 0 & i = 1, \dots, n \\ z_i a_i & = & 0 & i = 1, \dots, n \\ y_i & \geq & 0 & i = 1, \dots, n \\ z_i & \geq & 0 & i = 1, \dots, n \\ a_i & \geq & 0 & i = 1, \dots, n \\ a_i & \leq & 1 & i = 1, \dots, n \\ \sum_{i=1}^{n} \xi_i \mathbb{E}S_i a_i & = & -k + \sum_{i=1}^{n} P_i - \sum_{i=1}^{n} \mathbb{E}S_i. & \end{array}$$

We now proceed to solve this system. Let us first temporarily ignore the last KKT condition involving the sum and let us select a specific policy i: we now consider three potential situations

1. If we assume $z_i > 0$, the third and second sets of KKT conditions imply successively that $a_i = 0$ and $y_i = 0$, and the first set of KKT conditions further gives

   $$z_i = \lambda \xi_i \mathbb{E}S_i - 2\mathbb{V}arS_i.$$

   Since we must also have $z_i \geq 0$, we see that this situation can only happen if

   $$\lambda \xi_i \mathbb{E}S_i \geq 2\mathbb{V}arS_i \Leftrightarrow \frac{\lambda \xi_i \mathbb{E}S_i}{2\mathbb{V}arS_i} \geq 1$$

   (and in that case all the other KKT conditions –ignoring the last one– are satisfied)

2. Similarly, if we suppose that $y_i > 0$, the second and third sets of KKT conditions will successively guarantee that $a_i = 1$ and $z_i = 0$, which will imply

   $$y_i = -\lambda \xi_i \mathbb{E}S_i$$

   which, since we must have $y_i \geq 0$, will only be possible when

   $$\frac{\lambda \xi_i \mathbb{E}S_i}{2\mathbb{V}arS_i} \leq 0$$

   (and in that case all the other KKT conditions –ignoring the last one– are satisfied)

3. Finally, considering the remaining situation where both $y_i = 0$ and $z_i = 0$, we see that the first set of KKT conditions implies that

$$a_i = 1 - \frac{\lambda \xi_i \mathbb{E} S_i}{2 \mathbb{V}ar S_i},$$

which will be acceptable if and only if

$$0 \leq \frac{\lambda \xi_i \mathbb{E} S_i}{2 \mathbb{V}ar S_i} \leq 1$$

(and in that case all the other KKT conditions –ignoring the last one– are satisfied)

The situation is now quite clear: in all three cases, the crucial quantity

$$\phi_i = \frac{\lambda \xi_i \mathbb{E} S_i}{\mathbb{V}ar S_i}$$

has a specific range ($\phi_i \geq 1$ in the first case, $\phi_i \leq 0$ in the second one and $0 \leq \phi_i \leq 1$ in the last one), and one can easily check that the following formula gives perfect description for the three cases

$$a_i = \min\bigl(1, \max(0, 1 - \phi_i)\bigr).$$

We observe that the optimal value for the a_i variables is a function of ϕ_i alone, which itself depends only on the problem data and the multiplier λ. This dependence can be illustrated by the following two graphs (depending on whether ξ_i is positive or negative, a_i being identically equal to zero whenever $\xi_i = 0$), where the marked value on the λ-axis is equal to $\lambda_i^* = \frac{\mathbb{V}ar S_i}{\xi_i \mathbb{E} S_i}$, which corresponds to $\phi_i = 1$:

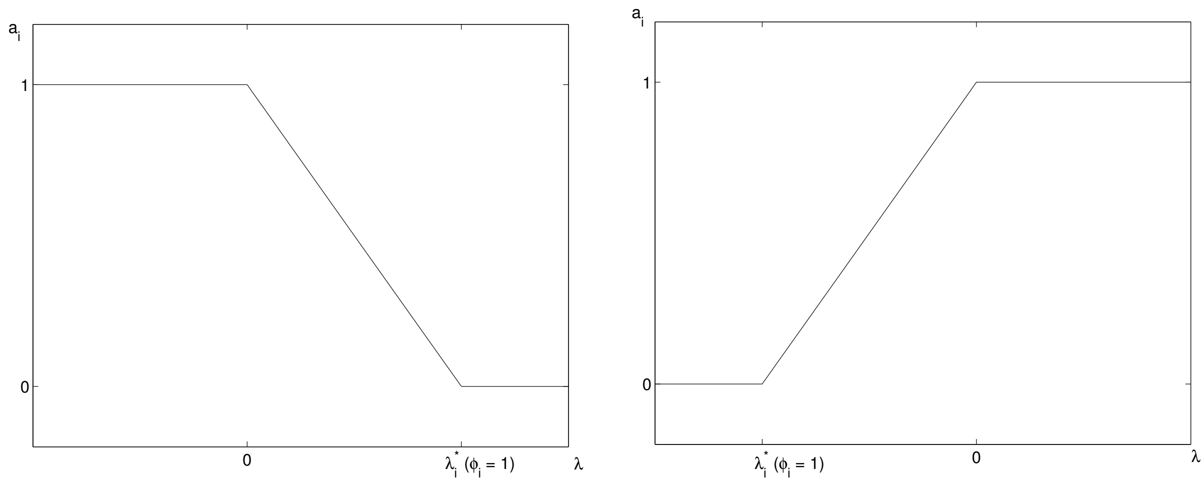

Figure 5.1: Dependence of a_i with respect to λ (left: when $\xi_i > 0$, right: when $\xi_i < 0$)

The last issue we have to settle pertains to the choice of λ and the last KKT equation: we first observe that since we have $0 \leq a_i \leq 1$, its left-hand side $\sum_{i=1}^{n} \xi_i \mathbb{E} S_i a_i$ can *a priori* vary between the following two bounds

$$B^- = \sum_{i | \xi_i < 0} \xi_i \mathbb{E} S_i \leq \sum_{i=1}^{n} \xi_i \mathbb{E} S_i a_i \leq \sum_{i | \xi_i > 0} \xi_i \mathbb{E} S_i = B^+.$$

We will now show that a suitable choice of λ can attain any value in that range. Assume first that λ takes a positive value: every ϕ_i such that $\xi_i < 0$ will be negative, so that the corresponding a_i will be one. Moreover, if that value of λ is sufficiently large, every ϕ_i such that $\xi_i > 0$ will be greater than 1, implying a zero value for the corresponding a_i. This shows that our left-hand side quantity attains its lower bound B^- in this situation.

If we start to decrease λ towards zero, each positive ϕ_i will also decrease, so that at some point each of the corresponding a_i will start to continuously increase (along with the total left-hand side), until they all become equal to one (in the meantime, every a_i corresponding to a negative ξ_i stayed equal to one). At this point, our left-hand side quantity is equal to $B^+ + B^- = \sum_{i=1}^n \xi_i \mathbb{E} S_i$. Decreasing further λ into the negative domain, we see that each a_i corresponding to a negative ξ_i start to decrease (while the other a_i stay equal to one), implying a continuous increase of the total left-hand side, until all of these a_i become zero: at this point, our left-hand side attains its upper bound B^+ and will keep that value for all smaller values of λ.

We deduce from this result that our KKT system of equations admits a solution if and only if its right-hand side lies between B^- and B^+, i.e. when

$$B^- = \sum_{i|\xi_i<0} \xi_i \mathbb{E} S_i \leq -k + \sum_{i=1}^n P_i - \sum_{i=1}^n \mathbb{E} S_i \leq \sum_{i|\xi_i>0} \xi_i \mathbb{E} S_i = B^+.$$

which is equivalent to

$$\sum_{i=1}^n P_i - \sum_{i=1}^n \mathbb{E} S_i - \sum_{i|\xi_i>0} \xi_i \mathbb{E} S_i \leq k \leq \sum_{i=1}^n P_i - \sum_{i=1}^n \mathbb{E} S_i - \sum_{i|\xi_i<0} \xi_i \mathbb{E} S_i \tag{5.2}$$

We have the following interpretations :

1. $\sum_{i=1}^n P_i - \sum_{i=1}^n \mathbb{E} S_i$ corresponds to the margin of the insurer before any cession to the reinsurer.
2. $\sum_{i|\xi_i>0} \xi_i \mathbb{E} S_i$ corresponds to the maximal positive margin of the reinsurer.
3. $\sum_{i|\xi_i<0} \xi_i \mathbb{E} S_i$ corresponds to the maximal negative margin of the reinsurer.

We can now comment on the actuarial intuition behind (5.2) :

1. If the first inequality in (5.2) is not satisfied, it means that that the cost of reinsurance cannot decrease the margin as low as the required $\mathbb{E} Z(\boldsymbol{a}) = k$. The problem then has no solution.
2. If the second inequality in (5.2) is not satisfied, it means that the gain provided by the reinsurance cover is not enough to reach the required $\mathbb{E} Z(\boldsymbol{a}) = k$. The problem then has no solution.

Combining now our results leads to de Finetti's formula

$$a_i = \min\big(1, \max(0, 1 - \frac{\lambda \xi_i \mathbb{E} S_i}{2 \mathbb{V}ar S_i})\big) \tag{5.3}$$

where λ is computed by plugging (5.3) into (5.1) (which is possible as long as (5.2) is satisfied). The fact that the KKT system of equations is both necessary and sufficient shows that it indeed provides the optimal value for the cession percentages in all cases.

6 Variable Quota Share Reinsurance

In practice, one does not apply a different cession percentage to each risk in the portfolio. In fact, the portfolio is partitioned into several segments for which an individual cession percentage is applied. This situation is called variable quota share reinsurance.

Let us assume that there are m such segments in a portfolio of $n = n_1 + \cdots + n_m$ independent risks. The result becomes

$$Z(\boldsymbol{a}) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} (P_i^j - (1 + \xi_i^j) a_j \mathbb{E} S_i^j - (1 - a_j) S_i^j).$$

Using the same arguments as above we are able to show that the optimal cession rates are given by

$$a_j = \min \left(1, \max \left(0, 1 - \frac{\lambda \sum\limits_{i=1}^{n_j} \xi_i^j \mathbb{E} S_i^j}{2 \sum\limits_{i=1}^{n_j} \mathbb{V}ar S_i^j} \right) \right)$$

where λ is a constant given by $\mathbb{E} Z(\boldsymbol{a}) = k$. There will be a solution when

$$\sum_{j=1}^{m} \sum_{i=1}^{n_j} (P_i^j - \mathbb{E} S_i^j) - \sum_{i,j | \xi_i^j > 0} \xi_i^j \mathbb{E} S_i^j \leq k \leq \sum_{j=1}^{m} \sum_{i=1}^{n_j} (P_i^j - \mathbb{E} S_i^j) - \sum_{i,j | \xi_i^j < 0} \xi_i^j \mathbb{E} S_i^j \tag{6.1}$$

7 Surplus Reinsurance with Table of Lines

For surplus reinsurance, it often happens in practice that the portfolio is partitioned into segments where the same line is applied. This situation is called surplus reinsurance with table of lines.

Defining $Q_i = \frac{S_i}{SI_i}$, the relative loss of risk i, we obtain the optimal table of lines as follows:

$$R_j = 1 - \frac{\lambda \sum\limits_{i=1}^{n_j} \xi_i^j \mathbb{E} Q_i^j}{2 \sum\limits_{i=1}^{n_j} \mathbb{V}ar Q_i^j}$$

where λ is a constant given by $\mathbb{E} Z(\boldsymbol{a}) = k$. There will be a solution if condition (6.1) is satisfied.

8 Numerical Example

Let us consider a portfolio with 4 risks having the following characteristics :

Risk	1	2	3	4
SI	100	200	100	200
$\mathbb{E}S$	15	50	35	90
$\mathbb{V}arS$	1500	6000	1500	6000
P	18.75	62.5	43.75	112.5
ξ	0.25	0.25	0.25	0.25

Table 8.1: Characteristics of the portfolio

We have the following optimality results, computed for two different levels of expected result and for different reinsurance structures :

- Quota share reinsurance
- Variable quota share reinsurance, where risks 1 and 2 have common cession rate and risks 3 and 4 have common cession rate
- Surplus reinsurance
- Surplus reinsurance with table of lines, where risks 1 and 2 have common line and risks 3 and 4 have common line
- Proportional reinsurance where the cession rate is freely chosen for each risk.

	$k = 20$	$k = 40$
Quota Share	2659.28	10637.12
Variable Quota Share	2418.14	9674.56
Surplus	2666.67	10408.16
Table of Lines	2400.00	9821.01
Proportional	2341.46	9652.17

Table 8.2: Optimality results

Proportional reinsurance is optimal : this is logical because the corresponding minimum is computed over the largest feasible region. Variable quota share is optimal compared to quota share and surplus with table of lines is optimal compared to surplus for the same reason.

Let us now proceed to the comparison of quota share and surplus reinsurance : for $k = 20$ quota share is superior to surplus, while the opposite is true for $k = 40$. However, for $k = 20$, surplus with table of lines is superior to variable quota share, but the opposite is again true for $k = 40$.

This numerical example shows that there exists no general rule asserting superiority of either quota share-type or surplus-type reinsurance above the other one.

References

[1] de Finetti, B. (1940), Il problema dei pieni, Giorn. Inst. Ital. Attuari, 11, 1–88.

[2] Bühlmann, H. (1970), Mathematical Methods in Risk Theory, Springer.

[3] Nocedal, J. and Wright S.J. (1999), Numerical Optimization, Springer.